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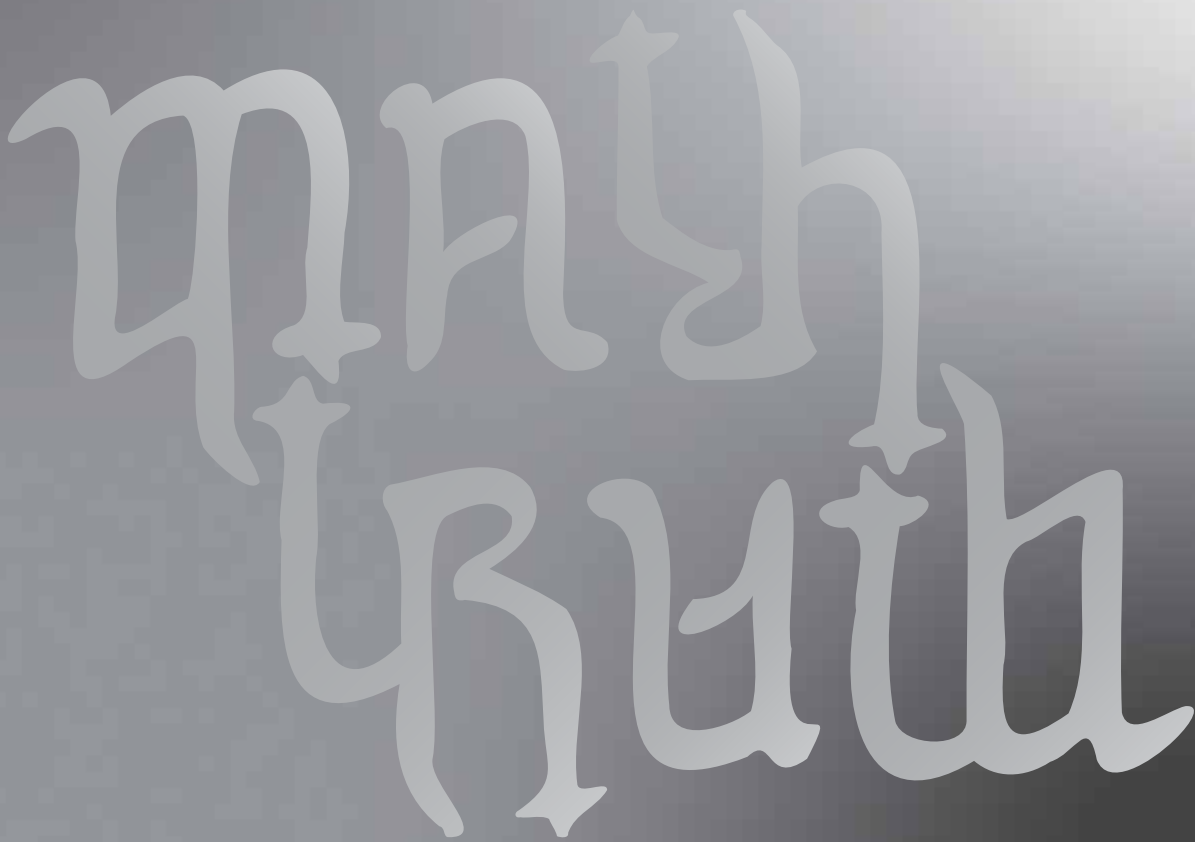
A publication of Azim Premji University together  
with Community Mathematics Centre, Rishi Valley



# Azim Premji University At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

ISSN 2582-1873



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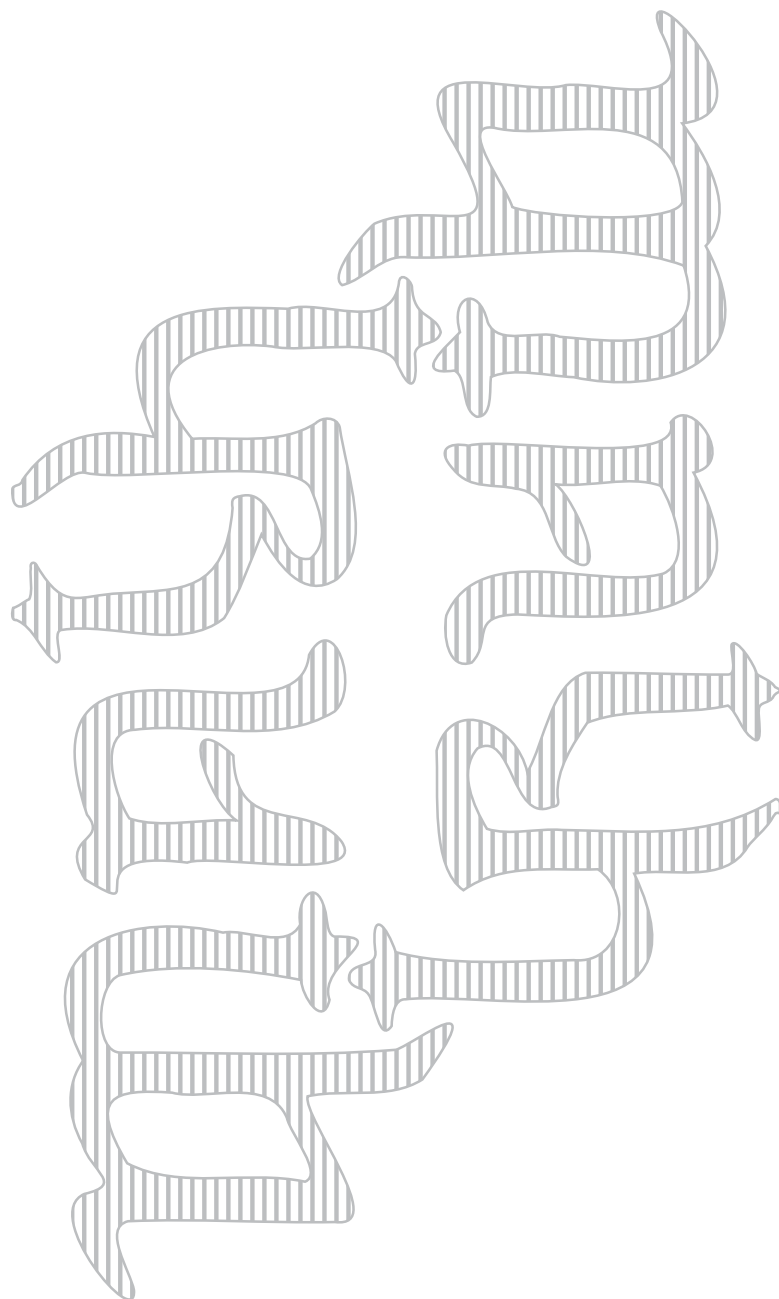
- » Exploring Geometrical Constructions – The GeoGebra Way

**PULLOUT**  
LOGIC, REASONING AND PROOF

RNI No.: KARENG/2018/76619

*The cover says it all: Mathematics and Truth are reflections of each other. When we prove something in mathematics, there can be no two ways about our argument.*

Read some fine proofs in the July 2023 issue of At Right Angles to enhance your understanding of a mathematical proof.



**Designs Courtesy**

Dr. Punya Mishra, Associate Dean of Scholarship & Innovation at Mary Lou Fulton Teachers College, Arizona State University. You can find him at [punyamishra.com](http://punyamishra.com)

# From the Editor's Desk . . .

Would you agree with me that the year is rushing past? We are already past the half-way mark of 2023! I'm reminded of the poem 'Leisure' by William Henry Davies which many of us would have studied in school... *No time to stand beneath the boughs and stare as long as sheep or cows.* We at AtRiA thought that we would give our readers an opportunity to do just that- stare at the cover of the July issue until Math and Truth emerged from it. This unique ambigram designed by Punya Mishra is itself a Proof Without Words, a teaser for the simply elegant proofs in several of the articles this time. We start with *Two New Proofs of the Pythagorean Theorem*, it's exciting to see young mathematicians thinking of ways to circumvent Circular Reasoning. Moshe Stupel and David Fraivert provide us with *Three Different Proofs for the Same Task* and Ankush Parcha, Toyesh Prakash, Ritam Sinha and Arnabi Saha all share results proved with various techniques, in articles and short fillers. Best of all, Padmapriya Shirali's PullOut is on *Proof* and how to get younger students to apply their reasoning and logic!

There is plenty for the math pedagogue in this issue- we have an account of a *Maths Mela- A Unique Measurement Fair in a School* by Ram Kumar Saroj. There's also an engaging classroom conversation in *What is the Square Root of 3*. Narrated by Rupesh Gesota, it talks about the meandering route to understanding- why take detours? Why understand processes? How to get students to explore? And you will find plenty of material for this in A Ramachandran's 3-Digit Numbers, in two articles on *Divisibility Rules*, in *Alternating Sums of Odd Numbers* and in Tech Space where Jonaki Ghosh talks about exploring *Geometric Constructions, the GeoGebra way*. Manisha Verma and Sandeep Diwakar wrap up the Classroom section with an article about their experience of teaching Probability.

Don't miss the account on *Fermat Numbers* by Yathiraj and the *Golden Conjecture* by Sasikumar. In Student Corner, our young students have their say- Shreya Mundhadha shares the magic in *Guess the Card* and Jayaditya reports on a *Magic Triangle* Incidentally, our Features article on Common Errors in Correlation, Causation and Association in Statistics is by a student- clearly one who looks before he leaps!

Don't forget that some of these articles are available only in the online edition- you will find the barcode on the Contents Page. *At Right Angles* holds a webinar on the third Wednesday of every month. Check out [https://www.youtube.com/watch?v=ZAs\\_3FTipzg](https://www.youtube.com/watch?v=ZAs_3FTipzg) based on the article [http://publications.azimpremjifoundation.org/3747/1/04%20COMPUTATIONAL%20THINKING\\_CLASSROOM.pdf](http://publications.azimpremjifoundation.org/3747/1/04%20COMPUTATIONAL%20THINKING_CLASSROOM.pdf) Listen to the experiences of two practitioners of Computational Thinking in Classrooms.

Happy ruminating!

**Sneha Titus**  
Associate Editor

# Opening Bracket . . .

## On Proof and Reasoning in the Teaching of Mathematics

Proof is central to the discipline of mathematics, and the exercise of looking for new proofs for known results, or proofs that embody an aesthetic element, is highly valued by mathematicians. G H Hardy writes in *A Mathematician's Apology* (which has been reviewed in this issue) about a method of proof that is very dear to mathematicians – ‘proof by contradiction’ or ‘reductio ad absurdum’:

*The proof is by reductio ad absurdum, and reductio ad absurdum, which Euclid loved so much, is one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.*

The proof he is referring to here is the one given by Euclid to prove the proposition that there are infinitely many prime numbers. It is a proposition for which new proofs continue to be found even in the present day. The same can be said about the Pythagorean theorem (and in this issue itself, we present a new and very novel proof found by two teenagers). For anyone who does not have a feeling for mathematics, this drive to find new proofs of a proposition that has already been proved dozens of times over would seem perfectly baffling! But this drive lies at the heart of mathematics.

As mathematics teachers, can we convey this feeling to students? It is important that we talk about it and attempt to show that proof in mathematics is worth studying and appreciating. We may fall short, because the obstacles are many, but it is important that we try.

What are the obstacles? Proof-writing requires a slowing down of our thought processes; one must go slowly and deliberately, making sure that there are no gaps in our reasoning; making sure that we do not miss out any possibilities. Thinking in this manner does not come naturally to us; we tend to think intuitively and inductively, generalising from specific instances (but not realising that we are doing so). It therefore becomes the task of the mathematics teacher to show that intuitive leaps and inductive thinking can lead us astray. But it is also the task of the same teacher to show the great value in problem-solving of intuitive leaps and inductive thinking! This only goes to demonstrate the great complexity of our task.

If we are to take this challenge seriously, then we must begin early and we must begin small. We must introduce students to proof-writing at the primary level, using appropriate contexts. Whole number arithmetic offers many possibilities in this regard. For example: *Why is the sum of two consecutive whole numbers always odd? Or: Why is the sum of three consecutive whole numbers always a multiple of 3? Or: Why is the sum of four consecutive whole numbers never a multiple of 4?* (The *Pull-out* describes more examples of this kind.) There are also puzzles and number games that can be used to devise appealing problems; e.g., cryptarithms; coin-weighting problems. Modular thinking is of great value here: dividing the problem into smaller chunks, and tackling each sub-problem completely. We see here the close relationship between problem-solving and proof-writing.

We may also regard proof-writing as an aspect of *communication*: i.e., communicating one's reasoning and thought processes precisely, accurately, and without ambiguity. This reveals yet another facet of proof-writing: its close relationship with language and therefore with thinking itself.

It will be good for mathematics teachers to come together and work out ways to make the writing of proofs an essential and enjoyable part of the teaching-learning of mathematics.

**Shailesh Shirali**

## Chief Editor

### Shailesh Shirali

Sahyadri School KFI and  
Community Mathematics Centre,  
Rishi Valley School KFI  
shailesh.shirali@gmail.com

## Associate Editor

### Sneha Titus

Azim Premji University,  
Survey No. 66, Burugunte Village,  
Bikkanahalli Main Road, Sarjapura,  
Bengaluru – 562 125  
sneha.titus@azimpremjifoundation.org

## Editorial Committee

### A. Ramachandran

Formerly of Rishi Valley School KFI  
archandran.53@gmail.com

### Ashok Prasad

Azim Premji Foundation for Development  
Garhwal, Uttarakhand  
ashok.prasad@azimpremjifoundation.org

### Hriday Kant Dewan

Azim Premji University  
hardy@azimpremjifoundation.org

### Jonaki B Ghosh

Lady Shri Ram College for Women  
University of Delhi, Delhi  
jonakibghosh@gmail.com

### Mohammed Umar

Azim Premji Foundation for Development  
Rajsamand, Rajasthan  
mohammed.umar@azimpremjifoundation.org

### K Subramaniam

Homi Bhabha Centre For  
Science Education, Tata Institute of  
Fundamental Research, Mumbai  
subra@hbcse.tifr.res.in

### Padmapriya Shirali

Sahyadri School, KFI  
padmapriya.shirali@gmail.com

### Prithwjit De

Homi Bhabha Centre For  
Science Education, Tata Institute of  
Fundamental Research, Mumbai  
de.prithwjit@gmail.com

### Sandeep Diwakar

Azim Premji Foundation for Development  
Bhopal, Madhya Pradesh  
sandeep.diwakar@azimpremjifoundation.org

### Shashidhar Jagadeeshan

Centre for Learning, Bangalore  
jshashidhar@gmail.com

### Sudheesh Venkatesh

Chief Communications Officer  
& Managing Editor,  
Azim Premji Foundation  
sudheesh.venkatesh@azimpremjifoundation.org

### Swati Sircar

Azim Premji University  
swati.sircar@azimpremjifoundation.org

## Editorial Office

The Editor, Azim Premji University  
Survey No. 66, Burugunte Village,  
Bikkanahalli Main Road, Sarjapura,  
Bengaluru – 562 125  
Phone: 080-66144900  
Fax: 080-66144900  
Email: publications@apu.edu.in  
Website: www.azimpremjiuniversity.edu.in

## Publication Team

### Shantha K

Programme Manager  
shantha.k@azimpremjifoundation.org

### Shahanaz Begum

Associate  
shahanaz.begum@azimpremjifoundation.org

## Print

SCPL  
Bengaluru 560 062  
www.scpl.net

## Design

Zinc & Broccoli  
enquiry@zandb.in

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**At Right Angles** is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

## Contents

### Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

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### ClassRoom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

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Continue . . .

## TechSpace

This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well as enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such as dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

Jonaki Ghosh

- 68 ▶ **Exploring Geometrical Constructions – The GeoGebra Way**

## Review

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach

mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

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## PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

Padmapriya Shirali

- Logic, Reasoning and Proof**

## Online Articles

# Two New Proofs of the Pythagorean Theorem - Part I

SHAILESH SHIRALI

The Pythagorean theorem ('PT' for short) is easily the best known result in all of mathematics. What is less well-known is the fact that among all theorems in mathematics, it holds the "world record" for the number of different proofs. There is no other theorem that even comes close! (See [2] and [3].) In the book *The Pythagorean Proposition* [1] (published in 1940), the author Elisha S. Loomis lists as many as 370 different proofs of the theorem. Since that time, close to a century back, still more proofs have appeared.

We present two of these proofs in this two-part article. The first one is by two high-school teenagers, Calcea Johnson and Ne'Kiya Jackson, both at St. Mary's Academy in New Orleans, USA. It has not yet been published so what we present here has been gleaned from media reports on their proof; see [4], [5], [6] and [7]. The second is an adaptation of a proof [8] by Professor Kaushik Basu, a well-known World Bank economist; he describes the proof as "new and very long" but gives a poetic and eloquent justification for adding this proof to the long list of existing proofs. This will appear in Part II of the article.

The proof by Calcea Johnson & Ne'Kiya Jackson raises an extremely interesting question. *Can there be a proof of the PT based on trigonometry?* In [1], the author claims definitively and strongly that there cannot be such a proof. It turns out that Loomis was wrong in making this claim, and the proof by Johnson & Jackson is itself a counterexample! We will say more about this interesting debate below.

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*Keywords: Pythagorean theorem, Kaushik Basu, Calcea Johnson, Ne'Kiya Jackson, St. Mary's Academy, New Orleans, trigonometric proof*

### Proof by Calcea Johnson and Ne’Kiya Jackson

We start by posing a basic question: *Can there be a trigonometric proof of the PT?*

It is possible that by ‘trigonometric proof’ we have in mind the following ‘argument.’ Let  $\triangle ABC$  be given, right-angled at  $C$ . Using the usual symbols we must prove that  $c^2 = a^2 + b^2$ . Observe that  $\sin A = a/c$  and  $\cos A = b/c$ . Since  $\sin^2 A + \cos^2 A = 1$ , we obtain  $a^2/c^2 + b^2/c^2 = 1$ , or  $c^2 = a^2 + b^2$ . Hence proved!

But how do we know that  $\sin^2 A + \cos^2 A = 1$ ? By using the Pythagorean Theorem, of course! So we have got trapped into a circular argument here, which means that we have not actually proved anything.

But is there a way of proving the basic trigonometric identity  $\sin^2 \theta + \cos^2 \theta = 1$  without using the PT and using only the definitions of sine and cosine? Yes, there is: we simply mimic the proof-by-similar-triangles of the PT. Here are the details (Figure 1). Let  $\triangle ABC$  be right-angled at vertex  $C$ ; let its hypotenuse  $AB$  have unit length. Denote  $\angle ABC$  by  $\theta$ . Draw a perpendicular  $CD$  from  $C$  to  $AB$ . Then  $\angle ACD = \theta$ .

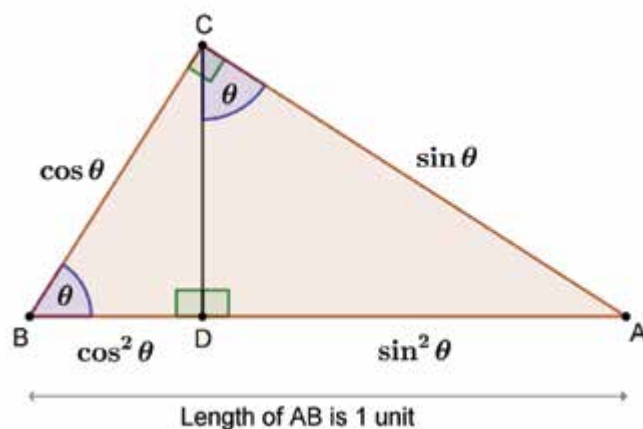


Figure 1. Proof of the trigonometric identity  $\sin^2 \theta + \cos^2 \theta = 1$  and the PT.

Since  $AB = 1$ , we get  $CA = \sin \theta$  and  $CB = \cos \theta$ , from the definitions of the trigonometric ratios. Since  $\triangle CBD$  is right-angled with hypotenuse  $CB$ , we have

$$\frac{BD}{CB} = \cos \theta, \quad \therefore BD = \cos^2 \theta. \quad (1)$$

Since  $\triangle CAD$  is right-angled with hypotenuse  $AC$ , we have

$$\frac{AD}{CA} = \sin \theta, \quad \therefore AD = \sin^2 \theta. \quad (2)$$

Since  $BD + AD = 1$ , we get  $\sin^2 \theta + \cos^2 \theta = 1$ , as required. And since  $\cos \theta = BC/AB$  and  $\sin \theta = AC/AB$ , we get  $CB^2 + CA^2 = AB^2$  or  $a^2 + b^2 = c^2$ , which is the PT. This way, we arrive at the statements of the PT and the basic trigonometric identity ( $\sin^2 \theta + \cos^2 \theta = 1$ ) at the same time. *Now there is no circularity of argument.*

With that preamble, let us study the proof given by Ms. Calcea Johnson and Ms. Ne’Kiya Jackson of New Orleans; see Figure 2.



Ne'Kiya Jackson, left, and Calcea Johnson recently presented their findings at the American Mathematical Society's south-eastern chapter's semi-annual meeting. Photograph: WWL-TV

Figure 2. Calcea Johnson and Ne'Kiya Jackson of St. Mary's Academy, New Orleans, USA. Credit: [https://www.theguardian.com/us-news/2023/mar/24/new-orleans-pythagoras-theorem-trigonometry-prove?CMP=oth\\_b-aplnews\\_d-1](https://www.theguardian.com/us-news/2023/mar/24/new-orleans-pythagoras-theorem-trigonometry-prove?CMP=oth_b-aplnews_d-1)

Here are the prerequisites needed to understand the proof.

- (1) The basic theorems concerning similar triangles (the results are well-known so we do not list them here);
- (2) The formula for the limiting sum of an infinite geometric progression with first term  $a$  and common ratio  $r$  between  $-1$  and  $1$ :

$$a + ar + ar^2 + ar^3 + \dots = \frac{a}{1 - r} \quad (-1 < r < 1);$$

- (3) The double-angle trigonometric identity  $\sin 2\theta = 2 \sin \theta \cdot \cos \theta$ .

Figure 3 shows the given  $\triangle ABC$  which is right-angled at  $C$ . We must show that  $c^2 = a^2 + b^2$ . We shall assume throughout that  $a < b$ ; equivalently, that  $\alpha < \beta$ .

Reflect  $\triangle ABC$  in  $AC$  to give  $\triangle ADC$ ; then  $\angle BAD = 2\alpha$ . Since  $\alpha < \beta$ , it follows that  $\alpha < 45^\circ$ , so  $\angle BAD < 90^\circ$ . Draw ray  $AD$ . Draw a line through  $B$  perpendicular to  $AB$ . This line will meet ray  $AD$  at  $M$ , say. (The two lines meet since  $\angle BAM < 90^\circ$ .) Join  $BM$ . Now draw  $DE \perp BD$ , meeting  $BM$  at  $E$ ; draw  $EF \parallel BD$ , meeting ray  $AM$  at  $F$ ; draw  $FG \perp BD$ , meeting  $BM$  at  $G$ ; draw  $GH \parallel BD$ , meeting ray  $AM$  at  $H$ ; and so on . . . (see the 'etc' in Figure 3).

We shall now compute the lengths of  $BM$  and  $AM$ ; this yields the desired result. We shall feel free to make use of trigonometric identities — provided that those identities do not require the PT for their proofs.

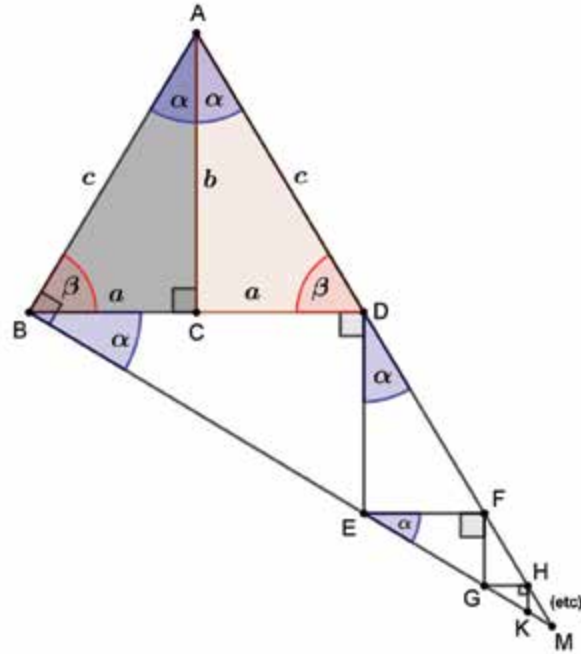


Figure 3. Proof of the PT by Calcea Johnson and Ne'Kiya Jackson.

Note the many triangles that are similar to  $\triangle ABC$ . We have:

$$\triangle ABC \sim \triangle BED \sim \triangle DFE \sim \triangle EGF \sim \triangle FGH \sim \triangle GKH \sim \dots, \quad (3)$$

as each of these triangles has angles  $\alpha, \beta, 90^\circ$ .

It follows that  $DE : BD : BE = a : b : c$ , so  $DE : 2a : BE = a : b : c$ , hence

$$BE = \frac{2ac}{b}, \quad DE = \frac{2a^2}{b}. \quad (4)$$

Next:  $EF : DE : DF = a : b : c$ , so  $EF : 2a^2/b : DF = a : b : c$ , hence

$$EF = \frac{2a^3}{b^2}, \quad DF = \frac{2a^2c}{b^2}. \quad (5)$$

Next:  $FG : EF : EG = a : b : c$ , so  $FG : 2a^3/b^2 : EG = a : b : c$ , hence

$$FG = \frac{2a^4}{b^3}, \quad EG = \frac{2a^3c}{b^3}. \quad (6)$$

Proceeding thus, we find that  $DF, FH, \dots$  form a geometric progression with common ratio  $a^2/b^2$ :

$$DF = c \cdot \frac{2a^2}{b^2}, \quad FH = c \cdot \frac{2a^4}{b^4}, \quad \dots \quad (7)$$

Similarly,  $BE, EG, GK, \dots$  form a geometric progression with the same common ratio,  $a^2/b^2$ :

$$BE = c \cdot \frac{2a}{b}, \quad EG = c \cdot \frac{2a^3}{b^3}, \quad GK = c \cdot \frac{2a^5}{b^5}, \quad \dots \quad (8)$$

We are now able to compute the total length of the segments  $DF, FH, \dots$  by summing an infinite geometric progression with common ratio less than 1:

$$\begin{aligned} DF + FH + \dots &= c \cdot \frac{2a^2}{b^2} \cdot \left( 1 + \frac{a^2}{b^2} + \frac{a^4}{b^4} + \frac{a^6}{b^6} + \dots \right) \\ &= c \cdot \frac{2a^2}{b^2} \cdot \frac{1}{1 - a^2/b^2} = \frac{2ca^2}{b^2 - a^2}, \end{aligned} \quad (9)$$

and therefore:

$$AM = c + \frac{2ca^2}{b^2 - a^2} = c \cdot \frac{a^2 + b^2}{b^2 - a^2}. \quad (10)$$

Similarly,

$$\begin{aligned} BM &= BE + EG + GK + \dots \\ &= \frac{2ac}{b} \cdot \left( 1 + \frac{a^2}{b^2} + \frac{a^4}{b^4} + \frac{a^6}{b^6} + \dots \right) \\ &= \frac{2ac}{b} \cdot \frac{1}{1 - a^2/b^2} = \frac{2abc}{b^2 - a^2}. \end{aligned} \quad (11)$$

It follows that

$$\frac{BM}{AM} = \left( \frac{2abc}{b^2 - a^2} \right) \div c \left( \frac{a^2 + b^2}{b^2 - a^2} \right) = \frac{2ab}{a^2 + b^2}. \quad (12)$$

On the other hand,  $BM/AM = \sin 2\alpha$ , as may be seen by observing the angles of the right-angled triangle  $ABM$ . Hence:

$$\sin 2\alpha = \frac{2ab}{a^2 + b^2}. \quad (13)$$

We now make use of the trigonometric identity  $\sin 2\alpha = 2 \sin \alpha \cos \alpha$ . From  $\triangle ABC$ , we get

$$\sin \alpha = \frac{a}{c}, \quad \cos \alpha = \frac{b}{c}, \quad \therefore \sin 2\alpha = \frac{2ab}{c^2}. \quad (14)$$

From the two relations for  $\sin 2\alpha$ , we conclude that

$$c^2 = a^2 + b^2, \quad (15)$$

and there we have it: the Pythagorean theorem! □

You may wonder about the identity  $\sin 2\alpha = 2 \sin \alpha \cos \alpha$ . Does it require the PT for its proof? No, it does not. (Homework for the reader!) So there is no circular reasoning in this proof.

You may also wonder what happens to the proof in the isosceles case when  $a = b$  (or  $\alpha = \beta$ ). We leave this part too to the reader.

*Closing remark.* Though the proof by Calcea Johnson and Ne'Kiya Jackson is long and involved, it is also subtle and deep. A truly wonderful achievement by the two teenagers.

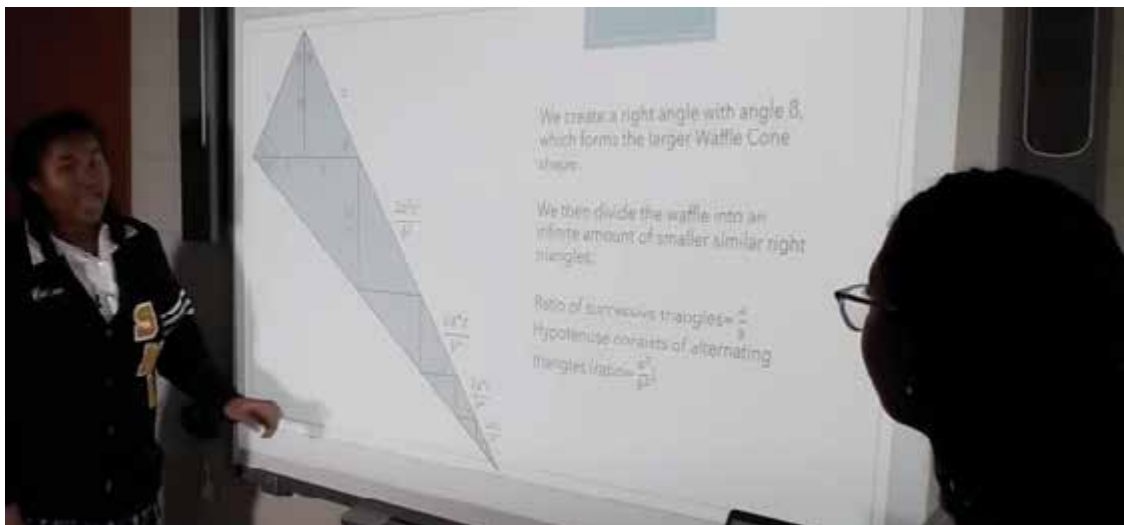


Figure 4. Calcea Johnson and Ne'Kiya Jackson presenting their proof.

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**SHAILESH SHIRALI** is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at [shailesh.shirali@gmail.com](mailto:shailesh.shirali@gmail.com).

# Common Errors in Interpretation of Correlation, Causation, and Association in Research

VEDANTH  
NANDIVADA

*“Don’t confuse correlation with causation.  
Almost all great records eventually dwindle.”*

– Charlie Munger

**D**uring research and data analysis, several questions arise regarding the relationships between variables. For example: In what way are the two variables related? Are they dependent on each other? Is there a cause-and-effect relationship? It is easy to misinterpret the relationships between variables from an experiment (Bewick et al., 2003). For example, consider the terms correlation, causation and association; they refer to the nature of relationships between variables. While correlation between two variables can imply association, it may not always lead to a causal effect of an independent variable on the dependent variable.

Though statistical formulae are objective, their interpretation is often subjective, so understanding the significance of relationships between variables is foundational for building statistical skills and communicating results in a scientific manner. In recent years, scholars have found that many research experiments could not be replicated, thereby questioning the credibility of research (Hope et al., 2021). Roughly half of all research in the natural and social science fields were considered false as they could not be replicated. This is referred to as

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*Keywords: Statistics, subjectivity, objectivity, association, cause-and-effect, errors, misinterpretation.*

the ‘replication crisis’ (Loannidis, 2005). This is partly due to inappropriate and inaccurate use of statistics in research. There is significant research on various types of errors and misinterpretations in using statistical techniques in research, and the urgent need for improving statistical training to address this crisis (Makin, Orban de Xivry, 2019).

This article summarizes the differences between *causation*, *correlation* and *association* between variables, common pitfalls in using the terminology through real world examples, and the statistical techniques to be used in future research.

Let us take a scenario where everyone in Section A of 6th grade played football for two hours a day and got 85% in their Math examination. All students in Section B of the same grade played football for half an hour each day, and everyone got 55% in the same exam. Soon, a student in Section B found out Section A’s secret to getting good grades; he excitedly told his classmates, “Guys, we must play football for more time and we will get better grades in our exam!” In the next exam, Section A again got 85%, while Section B got 30%, despite playing football for an increased number of hours. The students scratched their heads, wondering how the secret formula had failed them.

The above example illustrates the common misunderstandings between correlation and causation. Just because two events occurred together, it cannot be said that one happened because of the other.

## Definitions

It is important to understand the definitions of key statistical terms used in research before delving into the common errors in interpretation.

- An **independent variable** is the quantity that is changed or manipulated in a research experiment. Its changes in value do not depend on other variables in the experiment.
- The **dependent variable** is the quantity that is measured and the value generally depends on an independent variable.
- **Discrete** data is data that can only take a finite or countable set of values. Examples of this type of data are the number of children in a family: 0, 1, 2, 3, 4, ....
- **Continuous** data is data that can take an infinite number of values over a continuum. Examples include the height and weight of individuals.
- **Linear correlation** indicates the extent of *linear relationship* between two or more variables. The direction of the relationship between two variables is captured by the terms ‘positive’ or ‘negative’ that are attached to the word ‘correlation’. If there is a correlation, the pattern of correlation between two variables can be seen in a scatterplot. The *r*-value, often referred to as the Pearson correlation coefficient, measures the strength of the correlation and ranges between  $-1$  and  $1$ . The closer the value is to  $1$  or  $-1$ , the higher the strength of the correlation.
- **Association** indicates that two variables are dependent on one another. The terms association and dependence are used interchangeably to convey the message that a change in the independent variable *influences* a change in the dependent variable.
- **Causation** is a phenomenon where a change in a dependent variable is the *result* of a change in an independent variable.
- **Prediction** is the ability to predict (‘guess’) the value of the dependent variable for a given value of the independent variable given that an association has been confirmed between the variables. A regression model is used for the prediction.
- A **regression model** is a statistical technique to estimate a relationship between a dependent variable and one or more independent variables. In this paper, we will focus on linear regression models to determine the relationship between two variables. Linear regression models follow the equation  $y = \beta_0 + \beta_1 x$ , where  $y$  is the dependent variable,  $x$  is the independent variable,  $\beta_0$  is the value of the dependent variable when  $x$  is equal to  $0$ ,

and  $\beta_1$  is the change in the dependent variable for every unit change in the independent variable. Regression models are developed and enhanced by training and testing. During the training phase, the model uses the given  $x$  and  $y$  values to calibrate the  $\beta_0$  and  $\beta_1$  values in the model. During the testing phase, test values of  $x$  are inputted to the model and values of  $y$  are predicted by the model. These values are compared by the true value of  $y$  in the testing data to assess model accuracy. In this paper, when the term 'beta value' is mentioned, we refer to the  $\beta_1$  coefficient. The accuracy of the fit of the model and hence the degree of association is determined using a regression model by looking at its  $r^2$  value. This ranges between 0 and 1, with an  $r^2$  closer to 1 depicting a higher degree of accuracy of the prediction model.

- **The null hypothesis** suggests that there is **no** statistical relationship between two variables in an experiment. This hypothesis is assumed to be true until statistical analysis suggests otherwise. For example, in a science experiment to assess the effect of increasing concentrations of Vitamin C on plant shoot growth, the null hypothesis would be "Increasing Vitamin C concentration does not have a significant effect on plant shoot growth."
- **The alternate hypothesis** suggests that there **is** a statistical relationship between two variables in an experiment. This hypothesis is the opposite of the null hypothesis and is considered valid only when the null hypothesis has been discarded. Continuing the same example as above, in an experiment to assess the effect of increasing concentrations of Vitamin C on plant shoot growth, the alternate hypothesis would be "Increasing Vitamin C concentration has a significant effect on plant shoot growth."
- **The p-value** is a number that measures the evidence against the null hypothesis. More precisely, it tells us the probability of obtaining a result that is as bad as the result observed, assuming that the null

hypothesis is true. For example, consider an experiment designed to assess the effect of increasing concentrations of Vitamin C on plant shoot growth. As noted above, the null hypothesis would be "Increasing Vitamin C concentration does not have a significant effect on plant shoot growth." Suppose we obtain a  $p$ -value of 0.03. As this is less than 0.05, it means that the observed result is unlikely to have occurred by chance alone (assuming the null hypothesis). Therefore, we reject the null hypothesis in such a case.

Let us walk through three important common errors in correlation, association and causation.

### **Error 1 - "Correlation always implies causation"**

Linear correlation answers the following question: Is there a linear relationship between two or more variables? Take for instance, a study that showed a significant correlation between yearly chocolate consumption and the number of Nobel Laureates per country ( $r = 0.79$ ). This finding has led to the suggestion that an increased chocolate intake leads to an increase in the number of Nobel Laureates due to the cognitive effect of cocoa flavanols (Mourage et al., 2013). This incorrect assumption is predominantly due to the misunderstanding of the terms, 'correlation' and 'causation.' Correlation by itself does not have enough proof to infer causation. A strong correlation could simply be due to a sampling error or a random chance coincidence. Had a different sample been chosen, there is a possibility that a different  $r$  value could have been obtained, leading to a different conclusion. Hence, a high correlation does not always imply that a change in one variable truly *causes* the change in the other.

### **Avoiding the error**

1. *Research Question:* Are the two variables related?
2. *Statistical Technique:* If the variables are continuous, correlation analysis can be done using Pearson's or Spearman's coefficient.

3. *Statistical Analysis:* While analyzing the results, focus needs to be on the relationship between the variables, direction of the relationship, and strength of the relationship. If the  $r$  value is positive, there is a positive correlation. If the  $r$  value is close to 1, the correlation is strong.
4. *Recommended Terminology:* Key terms to be used in the interpretation of the correlation analysis are “correlated” or “related.” Terms that should not be used are “caused by. . .” or “influenced by. . .” For example, if you are studying a correlation between height and weight, you could summarize as “Height and weight are positively correlated.” Avoid the use of language such as “Weight seems to be caused by the height of the individual.” Ensure that the terminology does not refer to that of causation if no further analysis is performed beyond correlation (Makin, Orban de Xivry, 2019).

### **Error 2 - “Association always implies causation”**

Association conveys that there is a dependency between an independent variable and a dependent variable. Significance of the association between two variables can be determined by reviewing the  $p$ -value of the beta coefficients in a regression model. A positive beta coefficient depicts positive association whereas a negative beta coefficient depicts a negative association between variables. For example, several studies in recent decades have found an association between root canal treatment and protection against cardiovascular diseases. Nevertheless, there is not enough proof to deduce a causal relationship between the two variables (Jiménez-Sánchez et al., 2020).

Association by itself does not provide sufficient proof to infer causation. To infer causation, association needs to be backed up by consistency and specificity where the association is replicable in different studies, thus reducing its variability (Hill, 1965). Hence, a strong association does not necessarily imply that there is a causal relationship between two variables.

### **Avoiding the error**

1. *Research Question:* Are the two variables dependent on each other?
2. *Statistical Technique:* Linear regression analysis can be undertaken for understanding the dependency or association between a dependent variable and one or more independent variables. Variables can be discrete or continuous.
3. *Statistical Analysis:* While analyzing the results, focus needs to be on the significance of the influence of one variable over the other by verifying that the  $p$ -value is less than 0.05 given that the confidence interval is set at 95%. This indicates that for 95% of the experiments, the result falls under the alternate hypothesis.
4. *Recommended Terminology:* Key terms to be used in the interpretation of regression analysis are “factors were influenced” or “factors were dependent” or “factors were associated.” Terms that should not be used are “factor caused” or “causal analysis.” For example, if you are studying the influence of parental income on nutrition of children using regression analysis, you could say “It was observed that income influences children’s nutrition” or “Income and nutrition were found to be associated.” Avoid the use of language such as “Malnutrition seems to be caused by the parental income of the students.” (Nandivada, Gurtoo, in communication.)

### **Error 3 - “Prediction is the same as causation”**

Regression models have two predominant purposes which are often confused with one another. One is prediction, and the other is causation. Prediction involves deriving a formula based on the observed independent and dependent variable values in a training set in a study. Using the training set and the regression model, dependent variables can be predicted by inputting new values of the independent variable in the prediction model.

Prediction does not always imply causation. Causation can also be determined using a regression model by understanding whether an independent variable causes the change in the dependent variable (Allison, 1999). One important requirement to determine causation is to conduct randomized controlled experiments. This involves identifying two homogenous groups and treating one group as a control group and the other as a treatment group to compare the effect of treatment using various statistical tests (Gianicolo et al., 2020). To determine causation, there are multiple important steps which focus on ensuring both qualitative and quantitative proof that a change in the dependent variable is caused by a change in the independent variable in addition to determining an associative effect (Hill, 1965).

While a regression model can be used to predict a dependent variable, causation cannot be implied till additional statistical methods and analysis are accompanied.

### Avoiding the error

1. *Research Question:* Can the relationship between two variables be predicted? OR: Is there a cause-and-effect relationship between variables?
2. Statistical Technique:
  - *Prediction:* Regression analysis can be used to build a predictive model using a training data set. Variables can be discrete or continuous.
  - *Causation:* Use of control study and establishing two treatment groups will help in collecting the data for building a causation. Regression analysis and other probabilistic models can be developed to determine the causation if controlled randomized experiments are conducted to test for causality of a treatment.
3. Statistical Analysis:
  - *Prediction:* While analyzing the results for prediction, focus needs to be on the prediction model and the factors that could influence the accuracy of the prediction of dependent variables. To increase the  $r^2$

value of the model, one can use moderately increased training data to train the model and moderately reduced testing data.

- *Causation:* To determine causation, it is important to ensure homogeneity across groups before subjecting one group to a treatment. Regression analysis and additional statistical tests can be used for confirming the causation.
4. Recommended Terminology:
- *Prediction:* Key terms to be used in prediction analysis are “model predicts the factors influencing...” or “factors were dependent...” or “factors were associated...” Terms that should not be used are “factor caused” or “causal analysis.” If you are building only a prediction model based on regression analysis for education and income, you could summarize this as “Income changes are dependent on education” or “Income changes can be predicted based on education.” Avoid the use of language such as “Education causes income changes” unless it is observed through causal analysis.
  - *Causation:* If controlled experiments are conducted on two homogenous groups and causation is proved through statistical tests, then the term ‘causation’ can be used.

### Conclusion

Though statistical formulae are objective, the interpretation is often subjective. Therefore, the interpretation of relationships between variables is foundational for building statistical skills and communicating results scientifically. Additionally, accurate understanding of statistical techniques is important to produce research that can be replicated. In this paper we have summarized three common statistical errors in research (there are other types of errors, of course, but they are much less common). We hope further attempts will be made by the research community to improve the understanding of common errors in interpreting statistical techniques in research and by the public at large.

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**VEDANTH NANDIVADA** is a grade 12 student who is passionate about Math and Statistics. He published multiple research articles in high school journals such as 'Parabola', at The University of Sydney, and 'The Journal of Emerging Investigators'. He believes that there is a need to bridge the gap in the foundational knowledge and understanding of statistical techniques among high school students across the world. He enjoys reading about Bayesian statistics and hopes to pursue a career involving data and its applications. He may be contacted at [ved.nandivada@gmail.com](mailto:ved.nandivada@gmail.com)

## Proof without words: Area of Circle = $\pi r^2$

First, divide a circle of radius  $r$  into  $n$  identical sectors. If  $n$  tends to infinity, then these sectors become triangles. So,

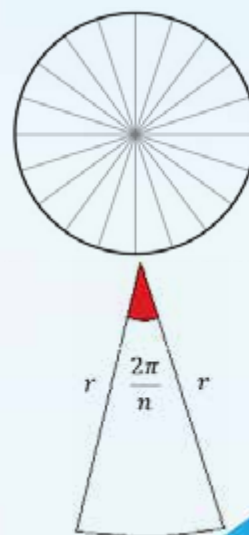
$$\text{Area of each triangle} = \frac{1}{2} r^2 \sin\left(\frac{2\pi}{n}\right)$$

If we add up the area of all the triangles, then we obtain

$$\sum \text{Area of triangle} = \frac{n}{2} r^2 \sin\left(\frac{2\pi}{n}\right)$$

Then, multiplying and dividing by  $\pi r^2$

$$\text{area of circle} = \pi r^2 \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}} = \pi r^2$$



Contributed by Ankush Kumar Parcha, Email id: [ankushparcha11@gmail.com](mailto:ankushparcha11@gmail.com)

# Detours – Unplanned Learning Opportunities

RUPESH GESOTA

Teachers have often seen that their carefully planned lessons take unexpected turns and end up not achieving their intended learning objectives. Here is a narrative from Rupesh Gesota who seizes such detours as opportunities for unintended learning and formative assessment.

Rupesh's plan was to see relationships between the side lengths in 30-60-90 Triangles. Without revealing this to his students, he told them to construct a couple of 30-60-90 Triangles with hypotenuses 10 cm and 6 cm respectively.

This is a sketch of their findings when he told them to measure the lengths of the remaining sides in both triangles. Rupesh's questions skillfully facilitated the following conclusion from the students.

**In both the triangles, the shortest side length is always half the longest one.**

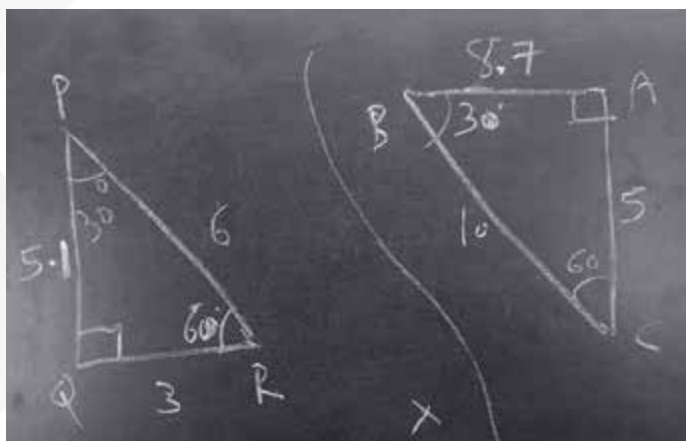


Image 1

*Keywords: Triangles, Ratios, Square-Roots, Exploration, Reasoning, Discussion*

He then told them to construct two more right triangles, with the hypotenuse as any whole number, but with angles other than 30° & 60° this time. They observed that in these cases the shortest side was not half the hypotenuse (the longest side). So, they said that this relation holds true only when the angles are 30-60-90.

Going back to the 30-60-90 triangle, he asked if there was any relation between the side opposite the 60° angle and the hypotenuse.

Since these side lengths were in decimals (5.1 and 8.7), it was naturally difficult for them to easily relate those to hypotenuse lengths.

He then guided them to the application of Pythagoras theorem, and they reached the step shown in Image 2. (Only the last part of the derivation, dictated by the students and written on the blackboard by Rupesh is shown.)



Image 2

Rupesh: How did you arrive at  $y = \frac{1.5x}{2}$ ?

Students: The square root of  $x^2$  is  $x$  and since the square root of 1 is 1 and the square root of 4 is 2, the square root of 3 is 1.5.

When asked to verify this, they started multiplying 1.5 by 1.5 using the standard procedure for multiplication of decimals. In Rupesh's words, *"I would have loved seeing them figure out this product mentally through reasoning. (Can you try that way?) I thought we would discuss this approach once they got the answer using this method, but then something else happened."*



Image 3: "I was now glad that they went by the standard multiplying method. They seemed puzzled / uncomfortable with their results. So, I told them to show their work on the board"

The students realised that both answers were incorrect since they were not between 1 and 4 as the square of 1.5 was expected to be. Rupesh understood that the students had difficulties with decimal multiplication and asked the students for another way to write 1.5.

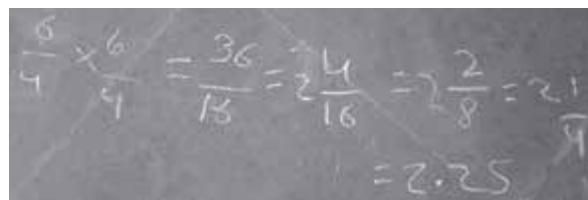


Image 4

They reasoned out that  $1.5 = 6/4$  because if 6 chapatis are given to 4 people, then each gets one and a half chappati, so  $6/4 = 1.5$ . Multiplication of fractions seemed to have been mastered by them and they also reasoned that  $1/4 = 0.25$  because 1 rupee = 100 paise, so a quarter means 25 paise. They were delighted with this answer 2.25 because it satisfied their expectation that  $1.5 \times 1.5$  needed to be between 1 and 4.

Further, something clicked for the boy who had worked out  $1.5 \times 1.5$  as 0.225 on the board; he went and corrected his answer saying that he had placed the decimal point incorrectly.

After some discussion, the rules of decimal multiplication were arrived at by the students who managed to correct their work based on the rules that they had arrived at. (For a complete account of the facilitation of this discussion, please visit Rupesh's blog, the link is given at the end of the article.) Rupesh gave them plenty of practice in decimal multiplication and when he was confident about their conceptual understanding of the algorithm, he went back to the original question:

Rupesh: What's the value of square root of 3? Is it 1.5?

Students (laughing): No, it will be between 1.5 and 2.

Rupesh: How do you know?

Students: Because we saw that 1.5 squared is 2.25 and 2 squared is 4.

Rupesh set them the task of finding the value of the square root of 3 for homework. And they left happily agreeing to this challenge.

When they arrived the next day, he asked them if they could find the value of the square root of 3. They confessed that they had tried but were not successful. They had arrived at the fact that 1.73 squared gave less than 3 and 1.74 squared gave more than 3 and they had concluded that the value of the square root of 3 was between 1.73 and 1.74.

Rupesh was glad that they worked with numbers having 2 digits after the decimal point, rather than just stopping at 1.7 and 1.8. But then he also wondered why they didn't go beyond that. He told them to list the numbers they had tried for the very first time and they wrote 1.6, 1.7, 1.8 and 1.9.

They also said that in the process of exploration, they had realised that the square root of 3 was between 1.7 and 1.8 and hence tried zooming in on that interval. Rupesh then asked them to write 1.71, 1.72, 1.73, but they stopped at 1.74, saying they did not try beyond this as 1.74 squared crossed 3... While he agreed with them, he asked them to continue listing beyond 1.74 & they wrote till 1.80. When they confessed that they did not know how to proceed, he drew their attention to the numbers on the left & told them to complete this list too - to write the numbers less than 1.6, and so they wrote till 1.1. [See Image 5]

Rupesh: How did you get these numbers between 1 and 2?

Students: By dividing the range into 10 parts.

Rupesh: Ok... and how did you get the numbers 1.71, 1.72, 1.73, etc.? I am asking this because I don't see them in the first column.

Students: We knew it's between 1.7 & 1.8, so we divided the range 1 to 2 into 100 parts now.

Rupesh circled the two numbers 1.7 and 1.8 when they said this and, while pointing at the circled portions, told them:

"So, can we say this second column is a kind of zoomed-in-picture between 1.7 and 1.8? Numbers which were present but not visible earlier have become visible now because you have divided the range into smaller, i.e., more parts. It's like you have kept a magnifying glass now on the two numbers 1.7 and 1.8."

He paused to help them understand this new perspective, then continued.

"So now you say that the answer, the number, is between 1.73 and 1.74. What can we do now?"

Students: We will divide the range further - into 1000 parts now - so that the numbers between 1.73 and 1.74 become visible. The student who said this also circled the pair 1.73 and 1.74.

They started listing from 1.731, 1.732 and so on till 1.740. So, he asked them what 1.740 represents. They said that it was the same as 1.74. So, then I told them to include another form of 1.73 too because they had circled this number too. Hearing this, the student wrote 1.730 above 1.731 in that column.

Now he drew their attention to the two circled pairs and the list of numbers next to each pair so that they could also actually see and not just visualize that the interval (1.7, 1.8) has the numbers from 1.70 to 1.80 in it and (1.73, 1.74) has the numbers from 1.730 to 1.740 in it.

The picture started looking like Image 5 in some time....

The squares of 1.731, 1.732 and 1.733 were calculated by them manually using the standard algorithm, but when it came to testing the squares of numbers in the next column, (those with more digits after the decimal place), then Rupesh became their assistant and helped them to get the squares of numbers which they wanted, with the help of his phone calculator.

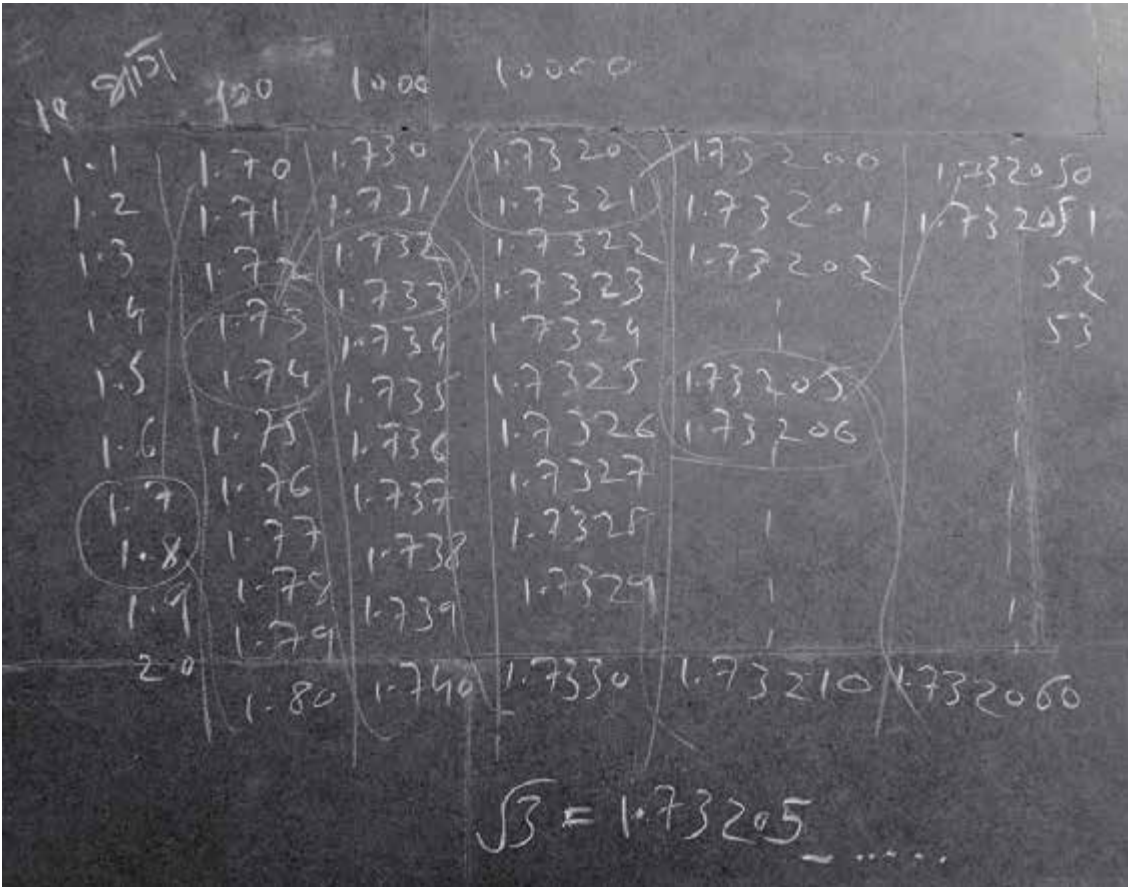


Image 5

Things had gone into auto-pilot mode now and they were enjoying this process, totally surprised as this hunting never seemed to stop, against their expectation. They said that they had never thought that square root of a number (that too such a small one like 3) will have so many digits.

Rupesh also shared with them that they need not list all the numbers in a column but could use dotted lines to indicate these. After some time, he stopped them and asked them what they thought about this process.

Students: Sir, it seems this is never going to stop.... We are just reaching closer and closer to the answer....

**Rupesh: How do you know this?**

Students: The square of the number comes out to be 2.9999.... or 3.0000 with a few other digits

after 9 and 0. And the number of 9's and 0's keep increasing....

He asked them if they could be very sure of at least some digits in the square root of 3?

They looked for a while in all the columns and noticed the growing & unchanging section of digits. As you can see in Image 5, they have written the value of square root of 3 as 1.73205.....

He asked them whether someone who told them that the square root of 3 equals 1.732 was correct.

They replied in the negative and explained that the answer is close to 1.732, but not equal to 1.732.

Looking at their facial expressions and body language, it was clear that this exercise was no less than an adventure ride for them. So now it was time to plug-in this correct value of root 3 into the expression they had arrived at (remember the original question).



Image 6

'x' and 'y' represent the respective lengths of hypotenuse and the side opposite to the angle measuring 60 degrees in the right-angled triangle.

They had observed that side length opposite to the angle measuring 30° is half the hypotenuse and the next attempt was to find the relation between hypotenuse and side opposite to the angle measuring 60°.

Now, they used the value of  $\sqrt{3}$  that they had calculated.

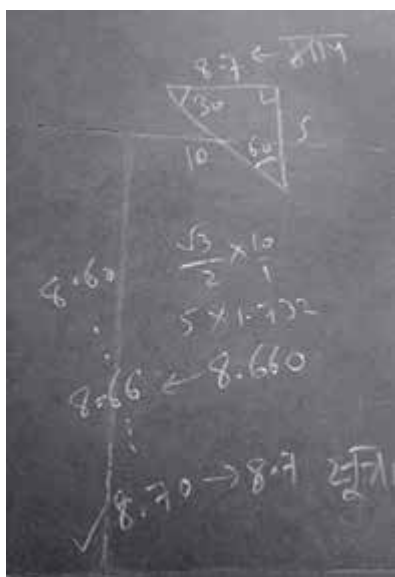


Image 7



Image 8

They were delighted to see that their measured lengths matched the lengths given by the formula.

We also discussed here about the round-off error, construction/ measurement error, resolution of measuring devices.

“Now we know why the square root 3 is written just as  $\sqrt{3}$  in textbooks!” they exclaimed.

So, what about the square root of 2?

They thought for a second and understood there can be many such numbers - square root of 5, square root of 7, etc., and I concluded this discussion by saying that such numbers are called as Irrational Numbers (of course with this thought in mind that this definition / explanation is not yet so precise and complete yet).

I moved to another topic, while giving them this assignment to find the value of square root of 2 (another irrational number which appears much in school mathematics) and they happily agreed to work on this.

## For a complete account of this discussion visit

<http://rupeshgesota.blogspot.com/2023/02/whats-value-of-square-root-of-3-part-1.html>

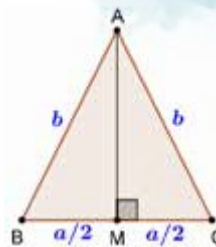
<https://rupeshgesota.blogspot.com/2023/02/whats-value-of-square-root-of-3-part-2.html?m=1>



**RUPESH GESOTA** is an engineer-turned-mathematics-educator. He loves to see the sparkle of understanding in the eyes of children, and he finds it inspiring to realise that he was part of this enlightenment process. He also loves working with their parents and teachers to make the process of Math education meaningful as well as joyful. To read more of his classroom stories check his blog [www.rupeshgesota.blogspot.com](http://www.rupeshgesota.blogspot.com). Rupesh may be reached through his website [www.rupeshgesota.weebly.com](http://www.rupeshgesota.weebly.com)

## AREA OF AN ISOSCELES TRIANGLE

We know that the area of a triangle is given by the formula  $= \frac{1}{2} \times \text{Base} \times \text{Height}$ . The formula requires you to find height and base. What if you know only the sides of an isosceles triangle? In this short note, we discover a formula to find the area of an isosceles triangle if we know its sides.



### Theorem.

In  $\triangle ABC$ , let  $AB = AC = b$ ;  $BC = a$ ; and perimeter  $p$ . Then:

$$\text{Area of } \triangle ABC = \frac{a}{4} \sqrt{p(2b - a)}.$$

**Proof:** Let  $AM \perp BC$ . Then  $\triangle ABM \cong \triangle ACM$ , so  $BM = CM = a/2$ . Hence

$$AM = \text{height} = \sqrt{b^2 - \frac{a^2}{4}} = \frac{1}{2} \sqrt{4b^2 - a^2} = \frac{1}{2} \sqrt{(2b + a)(2b - a)} = \frac{1}{2} \sqrt{p(2b - a)}.$$

Therefore:

$$\begin{aligned} \text{Area of } \triangle ABC &= \frac{1}{2} \times \text{base} \times \text{height} \\ &= \frac{1}{2} \times a \times \frac{1}{2} \sqrt{p(2b - a)} = \frac{a}{4} \sqrt{p(2b - a)}. \end{aligned}$$

**Illustration.** The area of the isosceles triangle whose sides are 5, 5, 8. Here  $b = 5$ ,  $a = 8$ ,  $p = 18$ . Hence

$$\text{Area} = \frac{8}{4} \sqrt{18(10 - 8)} = 2 \sqrt{36} = 12 \text{ square units.}$$

**Acknowledgment:** The author is grateful to Prof. B. N. Waphare and Prof. P. M. Avhad for encouraging him to send this finding to *At Right Angles*.

Vikram A. Ghule Eklayva Shikshan Sanstha's DamuAnna Madhyamik Vidyalaya, Bhilwade, Tal- Pathardi, Dist-Ahmednagar, Pin- 414102. E-mail: vikramghule2016@gmail.com.

# Maths Mela: A Unique Measurement Fair in a School

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**RAM KUMAR SAROJ**

School events such as a fair or celebration allow children to learn about matters which are more realistic and connect to daily life, in a manner which is beyond the scope of textbooks. In my learning and teaching experience, I encountered many events centred around science and other subjects but very few of mathematics. When talking about having such events in mathematics at least in the mainstream public education system, the number is almost zero. The learning experience for mathematics has not been very joyful and experiential as compared to others. When I heard about the unique concept of measurement fair, “MAPAN MELA”, it quite excited me.

My first experience of this mela in a primary school was indeed amazing. Later, I read an article describing the same experience and, while it motivated me to try out such a mela, I was clueless about how to organize it. Not only were there a limited number of activities, but I also started doubting if it was appropriate for a larger age group of 10-18 years or even for teachers. Later, I encountered the IGNOU chapter on measuring units[1]. This talked not just about measuring length, weight, area, and volume, but also about the layered conceptual complexity of the units and methods. I understood that my assumptions about my students’ understanding of these concepts could be unreal and that the fair could have very important objectives.

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*Keywords: Measurement, student’s context, joyful learning, data, measurement, estimation, reasoning*

- To encourage mathematics and mathematical thinking among students.
- To promote joyful mathematics learning experiences by connecting the concepts they had to learn to their daily life experiences.
- To encourage and nurture the approximation and estimation skills of learners using resources from their environment.

This helped me redesign and also add some activities which cover and excite children of all age groups. Now we call it both a Mathematics Fair and a Measurement Fair.

**Preparation:** Preparations for this fair are minimal in terms of time as well as resources. The material, which is all locally available, is listed along with the activities in the table below.

But I will explain the points which should be kept in mind for each activity. “Guessing” is at the center of all activities listed as the facilitator needs to ask the participants to guess the result for each measurement and then be allowed to check the result against the guess.

Measurement Fair - Registration Slip						
S.No.	Name of the activities	Guess				Resources required
1	Name					
2	Age	Years				
3	Weight		Kilos			
4	Length/Height		feet	inches	cm	tape measure
5	BMI					BMI chart and formula
6	Width		feet	inches	cm	tape measure
7	Foot length		cm	cm		tape measure
8	Palm length		cm	cm		inch tape
9	Body temperature		F	°C		thermometer or temperature gun
10	Pulse rate		bpm			stopwatch
11	Count per minute (grains)					a bowlful of pulses or cereals
12	Count in one breath					Count down from 1001
13	Breath-holding capacity		s			stopwatch
14	Water drops on a coin		Estimate		Actual	coin, dropper, a glass of water
15	Volume of a solid object		Estimate	ml	ml	measuring jar, beaker, putter, or any solid object
16	Change of Volume (Volume equivalence)		a bucket	how many mugs	how many glasses	glass, mug, cup
17	Palm area measurement		sq. cm			graph paper, pencil, colour
18	Footstep		first bottle	second bottle	third bottle	any three different solid objects such as bottles
19	Side measurement of Table/Black Board.		how many Bitta/palm length			
20	Weighing of 1 breath out of gas		g			Balloons, Small Weighing Machine

- **Measuring length and breadth of the body:**

It is good to mark a scale (generally, of 100cm to 180 cm) on the wall with the help of a pencil or pen. While measuring the height, participants should remove footwear. A small ruler (scale) should be used to press down the hair before the height is marked on the scale on the wall. It's good to go with the centimetre scale but the facilitator should know the foot and inch scale along with metre and centimetre. While measuring the width of the body, make sure the hands are fully open up to the middle finger and in a straight line with the shoulders. For accuracy, it's good to mark finger-tip to finger-tip on the wall and then measure the breadth of the participants. This avoids the curved path and the false increase when the tape is wrapped around the body.

- **Measuring weight:** Always measure without shoes and overclothes (sweater, coat, shawl etc.). This helps to get a precise value.

- **BMI Index:** Calculating Body Mass Index (BMI) using the measured weight and height. It is an indicator of good health and the right proportion of weight as per height. It's good for the facilitator to know how BMI gets calculated ( $BMI = \text{Weight}/\text{Height}^2$ ). The normal range is 18.5-25. Below 18.5, the person is underweight, those more than 25, the person is overweight, and those with BMI above 30 are considered obese. Facilitators should read more about BMI and know its limitations. As per current thought, BMI does not measure body fat directly, it should not be used as a diagnostic tool. Instead, BMI should be used as a measure to track weight status in populations and as a screening tool to identify potential weight problems in individuals [5].

- **Measuring body temperature:** This is also very easily done with temperature guns, but facilitators should discuss the unit of measurement and the conversion from Fahrenheit to Celsius and vice-versa. This is appropriate to discuss with 8-10 grade students.

- **Measuring palm and foot length:** It's good to mark the scale on a piece of chart paper and paste it on the ground or bench. Participants can measure the palm and foot length by putting them over it.

- **Counting in one breath:** This is a fun activity along with counting and checking number sense. Participants are asked to guess the number up to which they can count in one breath. Usually, they count from 1. After they are done, you can ask them to count from 1000, or in hundreds, or even in reverse order from a stated number.

- **Counting cereals:** This activity again checks counting and estimation skills in a fun way. There can be multiple participants who can check their answers against each other's counts.

- **Counting pulse rate:** First, the facilitator should know what is pulse rate and how it is related to the heartbeat. What is the pulse rate of a normal person? Where can we easily feel the pulse? Finding the pulse is an achievement for students and I have seen the magical sparkle in their eyes the moment they get it, especially the smaller ones. Let them take time to find it. Before they measure, let them ask the 'candidate' to relax and take a deep breath at least five times. Avoid measuring when they have just come running or doing heavy work of any kind. For a timer, a wall clock or hand/smartwatch, or mobile can be used. It is good to take the best of three measurements.

- **Holding the breath:** It is another health exercise that participants take as a challenge and that's fun. They do enjoy doing this. The result can be taken out of the best three. Good to use a mobile stopwatch or smartwatch. The facilitator is advised to do this test separately for each child and ensure that this does not turn into a dangerous competition.

- **Counting the water drops on a coin:** These are some of the exercises which overlap with science but here we focus on the mathematical part. As this activity is not so common, participants see a significant difference between

their guess and the actual number, and that amazes them. The use of multiple droppers and different coins is suggested when working with a larger group of participants. It's good to put this stall in a corner and restrict the number of visitors, otherwise, the guesses get influenced.

- **Measuring the volume of a solid object:**

The apparatus is specific - a measuring jar or beaker. The solid object could be anything but a paperweight or any small pebble which could be submerged easily in a mug or measuring beaker is preferred. This activity starts with giving the apparatus to participants after they guess the volume. If they have no idea about how to find the volume, leading questions and some instructions can be used to scaffold the measurement. If all else fails, a demonstration can be done in two ways: Fill a mug placed on a container (to catch the overflow) with water. Drop the solid object (whose volume needs to be found out) gently into the mug. Let it settle and then collect the water which has overflowed into the container. Pour it into the measuring cylinder and read the scale. That will be the approximate volume of the solid.

Another way is taking water into a measuring beaker and marking its level on the scale printed. Drop the solid so that it gets submerged fully and the water level rises. Read this new scale level. Subtract the previous reading from the second one. This gives the volume of the solid object.

- **Transformation of volume equivalence:**

This concept is related to volume transfer from one vessel to another. This adventure also starts with the question of how much or many of the small vessels will be equivalent to a large vessel. Let them guess and then let them check for themselves. This could be done with a bucket and mug or even a mug with small tea cups.

- **Measuring the area of the palm or any irregular 2D shape:** The object should be outlined over graph paper. (The palm could be dipped in watercolours and the print taken over graph paper.) Count the square boxes which are

completely or more than half inside the print area. Omit the squares which have more than half uncovered. The total count will give the approximate area of the palm in square units.

- **Measuring in footsteps:** This activity uses footsteps to measure lengths. This is one of the common local units to communicate the measurements of a piece of land in many cultures. In this activity, 2 -3 bottles may be kept as markers (or an "X" with chalk on the floor could be marked) at three points A, B and C. Participants first guess the distance in footsteps and then measure to verify.

- **Measuring with Palm:** This is another common method used in the villages. In this task, participants are asked to first guess the length of the table or window and then measure it. Facilitators should try to measure the lengths of different objects for different candidates.

- **Weighing the gas of one breath:** This is a task to measure the weight of gas breathed out at one time. This could be collected in a balloon. Do measure the weight of the empty balloon first by using a digital or conventional lightweight scale to which the balloon is tied with a string. After exhaling into the balloon, measure the weight again.

The list of activities can go long and we have many activities which certainly engage children and provide a learning opportunity, to think about, create and add to the list. The core idea remains the same, i.e., the activity is simple, very low-cost material is used and there is space for speculation and mathematical exploration. Activities from other subjects may be used but we explore the mathematical part of it. Here, the activities were centered around measurement but many other fun yet challenging tasks/games/puzzles like a picture puzzle, mind game reader, card game, fun with matchsticks, mathematical treasure hunts and many local popular games may be included as well. This ensures a good day-long activity and mathematics fair.

### **Do's and Don'ts during the fair preparation:**

1. It's good if the facilitators practise the activities among themselves for a day or two and also read and discuss the basic facts behind each of the activities.
2. Facilitators in each stall can be given logistical support.
3. Stalls should be spaced out so that students do not form a crowd.
4. Avoid crowding a stall so that students do not get influenced by other participants' estimates and answers.
5. It is good to have a couple of sets for each activity if the participant number is large.
6. A clear map with stall locations is needed at the entrance. Also, each stall should be labelled with bold letters.
7. The attached registration slip is mandatory for all participants to attend the fair. If not available, participants can use plain paper to note down their guesses and results for each activity.
8. To fill the response in the registration slip, a sketch pen or pencil may be used, with different colours for different stalls. A whitener might also be required to correct responses.
9. The registration slip is good to go with two prints per page. (Try back-back prints to save paper.)
10. Facilitators should be conscious of mentioning the unit while giving the final result to the participants.

Pre- and post-talk are very necessary and helpful to see how much they enjoyed and if they have come across any findings during this experience. Like length and breadth being almost the same and palm and foot length as well. As we practise and become aware, our capacity for counting increases. As a facilitator, I found this fair an opportunity to start discussions about measurement on an interesting note. In the conventional method, this part has been confusing and too focused on facts and

calculations. My focus was on the identification of local and standard units to measure different physical quantities and the conversion of units to their subunits. But this could provide a dialogic platform to explore questions such as the need for measurement. Why are standard measures used? Why are there sub-units of measurements and why are different units used for different objects and quantities? Starting from the concept of measuring length, areas, volume, capacity, mass, quantity, current, etc., may be explored based on the level of students. I found all of it very appropriate for all school students up to grade 12 along with teachers and parents as this all together helps to nurture the measuring skill and ability to choose appropriate measuring units based on the need of measuring a physical quantity. I also see that there is a lot of scope to go further and explore more with more advanced measuring tools such as vernier calliper, screw gauge, multimeter, sextant, etc.

The facilitators as per the capacity and scope of space and time may extend most of the activities into making a pool of data to understand data handling which would include representation and interpretation of data. They may, in a manner appropriate to the age and grade of the students, calculate central tendencies such as mean, mode, and median and their significance along with dispersion and deviations. All of these activities inside and outside provide interesting and contextual data to interact and learn important data concepts without going much into the textbooks or external data. But this has to be precisely planned with cues and questions to lead students into much more complex concepts in a more relatable and contextual manner. Findings may be put up near the stall itself and will be certainly interesting to the viewer.

Thus a math mela, while certainly a break from the routine of classroom mathematics, has extensive scope for learner engagement, introduction and teaching of more complex concepts. If the student will not come willingly to the math classroom, then why not take mathematics to the student?

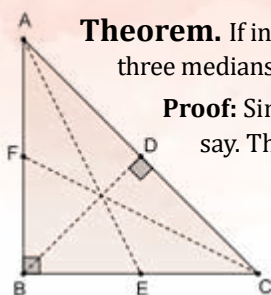
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**RAM KUMAR SAROJ** is an M.A. Education graduate from Azim Premji University, Bangalore. He has been working as a Teacher Educator at Eklavya Foundation, Shahpur, Betul since 2019, mostly with tribal school children and teachers using activity-based pedagogy for learning and teaching. He has been closely engaged with Eklavya's innovative approaches to support children in spaces such as Mohalla Learning Activity Centres (for classes 1-8), a Technology-Mediated Interactive Learning Platform Centres (TeMIL; a digital platform of learning for children of classes 5-8), and Chakmak Club (Adolescent Children's Club). This article is a work based on learning from these spaces. Ram Kumar may be contacted at [ram.ks@eklavya.in](mailto:ram.ks@eklavya.in)

## A RELATION BETWEEN THE MEDIANS OF AN ISOSCELES RIGHT-ANGLED TRIANGLE



**Theorem.** If in the isosceles, right-angled  $\triangle ABC$ ,  $\angle B = 90^\circ$ ,  $BD \perp AC$ , and  $BD, FC, AE$  are the three medians of  $\triangle ABC$ , then  $5BD^2 = AE^2 + FC^2$ .

**Proof:** Since  $BD$  is both a median and altitude, it follows that  $AB = BC = a$ , say. Then:

$$AE^2 = AB^2 + BE^2 = AB^2 + \frac{BC^2}{4} = a^2 + \frac{a^2}{4} = \frac{5a^2}{4}.$$

Similarly,

$$CF^2 = \frac{5a^2}{4}.$$

Hence

$$AE^2 + FC^2 = \frac{5a^2}{2}.$$

Next,  $BD = CD = \frac{AC}{2}$ , and  $AC^2 = a^2 + a^2 = 2a^2$ , so

$$BD^2 = \frac{AC^2}{4} = \frac{2a^2}{4} = \frac{a^2}{2},$$

and therefore

$$5BD^2 = \frac{5a^2}{2}.$$

Therefore,  $5BD^2 = AE^2 + FC^2$ , as claimed.

**Acknowledgment:** The author is grateful to Prof. B. N. Waphare and Prof. P. M. Avhad for encouraging him to send this finding to *At Right Angles*.

**Vikram A. Ghule** Eklavya Shikshan Sanstha's DamuAnna Madhymik Vidyalaya, Bhilwade,  
Tal- Pathardi, Dist-Ahmednagar, Pin-414102. E-mail: [vikramghule2016@gmail.com](mailto:vikramghule2016@gmail.com).

# Three Different Proofs for the Same Task

**MOSHE STUPEL &  
DAVID FRAIVERT**

A relatively simple and interesting geometric problem is presented, with three “wordless” style proofs attached to it: one in trigonometry and two in geometry. The goal of the task is for the learner to know how to verbally complete the proof according to the drawing and the mathematical expressions. By doing so, they will improve their visual ability to find proofs/solutions for different tasks.

A central part of teaching mathematics is writing proofs and solving problems. These two issues are related to each other. In each of them, the learner must present the method/way of proof or solution by using well-known theorems, attributes, or formulas, while their writing must be continuous and logical, so that the reader can understand the method of solution and confirm its correctness.

However, in reality it is not like that. The writers of mathematics articles, the teachers, as well as the students, usually skip listing some of the steps of the solution, because they think that a part of them are immediately understood and therefore there is no need to detail them. Such situations often leave the reader without an understanding of the solution or its correctness. On the other hand, there are cases where the solution/proof is excessively detailed, down to the smallest details. Such cases tire the reader, and they lose their attention and desire to continue reading.

## **Proof without words**

Especially in the last decade it is possible to see in various journals of mathematical education (some of them prestigious), both as a section and as a decoration, the presentation of “proofs without words” for various tasks. The proofs are usually presented with an illustration/drawing (sometimes with the addition of an auxiliary construction) as well as a listing of mathematical expressions and formulas, but without any words. The reader/student is expected to understand the steps of the proof and be able to formulate them verbally. From this activity, it is expected that the reader/learner will be able to develop their visual proof ability, when a large part of the tasks they face during their mathematics studies are based on tasks with illustrations and mathematical expressions [1-2].

“The illustration (drawing/diagram) and the mathematical expressions, are the lens through which the student understands the proof or identifies the argumentations to the solution.”

The “proof without words” is largely like a caricature in which a drawing appears with a certain message (sometimes with 2-5 words), that the viewer has to understand and absorb.

### Multiple solutions to the same geometric task

The studies in plane geometry constitute an important element of the studies of Mathematics, due to their contribution to the development of different channels for reasoning. It is important to note that the studies of plane geometry constitute a basis for the studies of trigonometry, solid geometry, analytical geometry, and they also have an important role in other branches of Mathematics. One of the principal objectives of the studies of mathematics is to impart to the students’ methods of reasoning that may assist them in other fields of learning and knowledge, and in solutions at higher levels. The meaning of “learning to think” means that the teacher of Mathematics must develop the students’ ability to apply information and perform analysis and synthesis at the adequate level of the basic properties, rules and theorems which has been taught at the earlier stages of the teaching process. The solution of different problems is one of the important means for reasoning development. Reasoning development is enhanced by solving problems using several different methods. Finding an additional method of solution using tools from the same mathematical field, and the more so when the tools are from a different field, promotes reasoning development and raises it to a higher level. Implementation of new knowledge in a new situation that leads to a shorter and simpler solution increases the pleasure and satisfaction that one obtains from one’s studies in Mathematics.

Integration of fields in problem solution opens a wider view on Mathematics for the students as a comprehensive subject that contains connections and interrelations between the different branches, reveals and accentuates its beauty.

The solution of a problem by a regular method leaves the students indifferent and without any particular reaction. However, a different solution to the same problem may elicit emotional excitement. A special and beautiful solution is surprising, unexpected and in most cases – short and simple [3-4].

The field of problems in plane geometry is a wide field of challenges where many solutions can be found both from this field and from adjacent or related fields.

### The problem

Given three identical squares adjacent to each other as seen in Figure 1. In the figure the angles are marked:  $\angle HCA = \alpha$ ,  $\angle HDA = \beta$ .

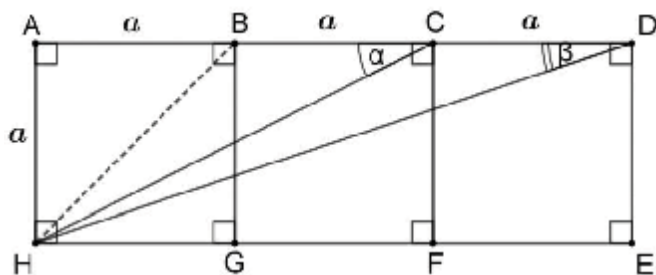


Figure 1

It must be proved that:  $\alpha + \beta = 45^\circ$

**Proof A: Using trigonometry (Figure 1)**

$$\tan \alpha = \frac{a}{2a} = \frac{1}{2}, \tan \beta = \frac{a}{3a} = \frac{1}{3}, \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \cdot \tan \beta},$$

$$\therefore \tan(\alpha + \beta) = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \times \frac{1}{3}} = 1 \Rightarrow \alpha + \beta = 45^\circ.$$

**A note**

In a classroom activity, it was found that students solved this task using a calculator, which is a legitimate way but less beautiful than the proof presented.

**Proof B: Using geometry (Figure 2)**

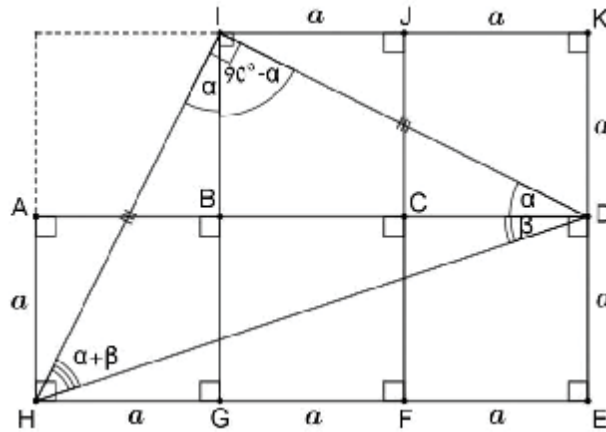


Figure 2

$$\triangle HID: \underbrace{IH=ID, \angle HID=90^\circ, \angle IHD = \angle IDH = \alpha + \beta,}_{\downarrow}$$

$$\therefore \alpha + \beta = 45^\circ.$$

**Proof C: Using geometry (Figure 3 and 4)**

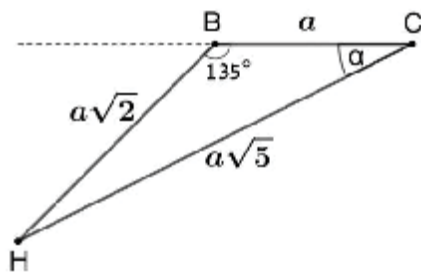


Figure 3

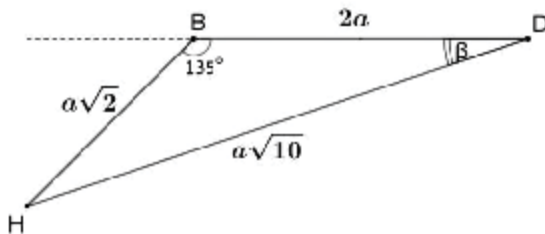


Figure 4

Consider  $\triangle HBC$  with sides  $a, \alpha\sqrt{2}, \alpha\sqrt{5}$  and  $\triangle HBD$  with sides  $\alpha\sqrt{2}, 2a, \alpha\sqrt{10}$ .

$$\frac{\alpha\sqrt{2}}{a} = \frac{2a}{\alpha\sqrt{2}} = \frac{\alpha\sqrt{10}}{\alpha\sqrt{5}} = \sqrt{2} \Rightarrow \triangle CBH \sim \triangle HBD \Rightarrow \alpha + \beta = 45^\circ.$$

## Notes

1. From the methodical point of view, the students should be expected to know how to complete the verbal reasons at each stage of the proof.
2. For this task there are additional proofs.

## Epilogue

From a comprehensive activity with students for teaching mathematics, understanding the reasoning for mathematical tasks presented through illustrations and mathematical expressions is of great importance in the training process. Students acquired another tool for their “mathematical toolbox,” and will be able to use it in their classrooms.

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**DR. MOSHE STUPEL** is Professor of Mathematics Education at Gordon College and Shaanan, College, Haifa, Israel. He has extensive experience in the study of dynamic properties using computerized technology and has published more than 200 articles in journals of Technology and Mathematics Education. He also deals with geometric constructions, finding different solutions to the same problem by combining different mathematical tools. He also has published two books and developed mathematical enrichment material. He may be contacted at [stupel@bezeqint.net](mailto:stupel@bezeqint.net)



**DR. DAVID FRAIVERT** is a Senior Lecturer and Researcher in the Department of Mathematics at Shaanan College in Haifa, Israel. His research interests are in geometry and mathematics education. He has published over forty papers in these fields, particularly regarding the discovery of new properties of geometric shapes and the development of a method for active learning that uses dynamic geometric software. Recently, his research has focused on a new topic in Euclidean geometry: the theory of a convex quadrilateral and a circle that forms Pascal points. He is the author of the book “A new theory in the geometry of a quadrilateral and a circle”. He may be contacted at [davidfraivert@gmail.com](mailto:davidfraivert@gmail.com).

# Divisibility by any Odd Number that is not a Multiple of 5

M.R. YADAV &  
NARESH KUMAR

Tests of divisibility exist for different divisors. In this article we present a test for divisibility by any odd number that is not a multiple of 5, i.e., divisibility by any number whose one's digit is 1, 3, 7 or 9. Examples of such numbers are 13, 19 and 27.

## Basic concepts and notation

- When a number  $n$  has one's digit 1, 3, 7 or 9, we can always find a multiple of  $n$  whose one's digit is 1 or 9. For example, if  $n = 13$ , we have  $7n = 91$ . We also have  $3n = 39$ .
- When  $p$  divides a number  $a$ , we denote this by  $p|a$ .
- If  $p|a$  and  $p|b$ , then  $p|(a \pm b)$ .
- The greatest common divisor (GCD) of  $a$  and  $b$  is denoted by  $d = (a, b)$ .
- If  $p|ab$  and  $(p, b) = 1$ , then  $p|a$ .

## The main result

Given an odd number  $p$  whose one's digit is 1, 3, 7 or 9, let  $pm = 10q \pm 1$  be a multiple of  $p$  whose one's digit is 1 or 9. Let  $a$  be the integer which we have to test for divisibility by  $p$ . Let  $b, c$  be integers such that  $0 \leq c \leq 9$  and  $a = 10b + c$ . Let  $d = b \pm qc$  (opposite sign as in the relation for  $pm$ ). Then  $p|a$  if and only if  $p|d$ .

*Keywords: Parity, divisibility*

**Proof: Case I.** When  $pm = 10q + 1$ .

Let  $a = a_n \dots a_2 a_1 a_0$  be an  $(n + 1)$ -digit number; then

$$\begin{aligned} a &= a_n 10^n + \dots + a_1 10^1 + a_0, \\ b &= a_n 10^{n-1} + \dots + a_2 10^1 + a_1, \\ c &= a_0, \end{aligned}$$

and  $d = b - qc$ . We must show that  $p|a \Leftrightarrow p|d$ .

Adding and subtracting  $10a_0q$  from  $a$ , we obtain

$$\begin{aligned} a &= (a_n 10^n + \dots + a_1 10^1 + a_0) + 10a_0q - 10a_0q \\ &= (a_n 10^n + \dots + a_1 10^1 - 10a_0q) + a_0(10q + 1). \end{aligned}$$

The quantity  $a_0(10q + 1)$  is divisible by  $p$ . Hence

$$p|a \Leftrightarrow p|(a_n 10^n + \dots + a_1 10^1 - 10a_0q).$$

Since  $(p, 10) = 1$ , it follows that

$$p|(a_n 10^n + \dots + a_1 10^1 - 10a_0q) \Leftrightarrow p|(a_n 10^{n-1} + \dots + a_1 - a_0q).$$

Now  $a_n 10^{n-1} + a_{n-1} 10^{n-2} + \dots + a_1 = b$  and  $a_0 = c$ . Hence

$$p|a \Leftrightarrow p|(b - cq),$$

i.e.,  $p|a \Leftrightarrow p|d$ , as claimed.

**Case II.** When  $P = pm = 10q - 1$ . Let  $a = a_n \dots a_2 a_1 a_0$  be an  $(n + 1)$ -digit number; then

$$\begin{aligned} a &= a_n 10^n + \dots + a_1 10^1 + a_0, \\ b &= a_n 10^{n-1} + \dots + a_2 10^1 + a_1, \\ c &= a_0, \end{aligned}$$

and  $d = b + qc$ . We must show that  $p|a \Leftrightarrow p|d$ .

Adding and subtracting  $10a_0q$  from  $a$ , we obtain

$$\begin{aligned} a &= (a_n 10^n + \dots + a_1 10^1 + a_0) + 10a_0q - 10a_0q \\ &= (a_n 10^n + \dots + a_1 10^1 + 10a_0q) - a_0(10q - 1). \end{aligned}$$

The quantity  $a_0(10q - 1)$  is divisible by  $p$ . Hence

$$p|a \Leftrightarrow p|(a_n 10^n + \dots + a_1 10^1 + 10a_0q).$$

Since  $(p, 10) = 1$ , it follows that

$$p|(a_n 10^n + \dots + a_1 10^1 + 10a_0q) \Leftrightarrow p|(a_n 10^{n-1} + \dots + a_1 + a_0q).$$

Now  $a_n 10^{n-1} + a_{n-1} 10^{n-2} + \dots + a_1 = b$  and  $a_0 = c$ . Hence

$$p \mid a \Leftrightarrow p \mid (b + cq),$$

i.e.,  $p \mid a \Leftrightarrow p \mid d$ , as claimed. ■

So, to check divisibility of  $a$  by  $p$ , we can instead check divisibility of  $d$  by  $p$ . The result may now be iterated, with  $d$  in place of  $a$ . Each iteration results in a smaller number to be checked. Ultimately, we are left with a small number for which divisibility by  $p$  can be checked mentally.

**Remark.** When there is a zero at the one's place in  $a$  (i.e., if  $a$  is divisible by 10), then we may first divide by 10 and then follow the above procedure. (See Problem 3.) This reduces the labour.

### Worked out examples

**Example 1.** Show that 2158 is divisible by 13.

**Solution:** Here  $a = 2158$ , so  $b = 215$ ,  $c = 8$ . Also,  $13 \times 3 = 39 = 4 \times 10 - 1$ , so  $q = 4$ .

Next,  $d' = 215 + (4 \times 8) = 247$ . We repeat the procedure with the new number.

$a = 247$ , so  $b = 24$ ,  $c = 7$ ,  $d' = 24 + (4 \times 7) = 52$ .

52 is divisible by 13, hence 2158 is divisible by 13. (Check:  $2158 \div 13 = 166$ .)

**Example 2.** Show that 65894 is not divisible by 17.

**Solution:** Here  $a = 65894$ , so  $b = 6589$ ,  $c = 4$ . Also,  $17 \times 3 = 51 = 5 \times 10 + 1$ , so  $q = 5$ .

So  $d' = 6589 - (5 \times 4) = 6569$ . We repeat the procedure with the new number.

$a = 6569$ , so  $b = 656$ ,  $c = 9$ . Hence  $d' = 656 - (5 \times 9) = 611$ . We repeat the procedure with the new number.

$a = 611$ , so  $b = 61$ ,  $c = 1$ . Hence  $d' = 61 - (5 \times 1) = 56$ .

56 is not divisible by 17, hence 65894 is not divisible by 17. (Check:  $65894 \div 17$  yields 3876, with remainder 2.)

**Example 3.** Show that 657894 is not divisible by 27.

**Solution:** Here  $a = 657894$ , so  $b = 65789$ ,  $c = 4$ . Also,  $27 \times 3 = 81 = 8 \times 10 + 1$ , so  $q = 8$ .

So  $d' = 65789 - (8 \times 4) = 65757$ . We repeat the procedure with the new number.

$a = 65757$ , so  $b = 6575$ ,  $c = 7$ . Hence  $d' = 6575 - (8 \times 7) = 6519$ . We repeat the procedure with the new number.

$a = 6519$ , so  $b = 651$ ,  $c = 9$ . Hence  $d' = 651 - (8 \times 9) = 579$ . We repeat the procedure with the new number.

$a = 579$ , so  $b = 57$ ,  $c = 9$ . Hence  $d' = 57 - (8 \times 9) = -15$ .

Since  $-15$  is not divisible by 27, we conclude that 657894 is not divisible by 27 either.

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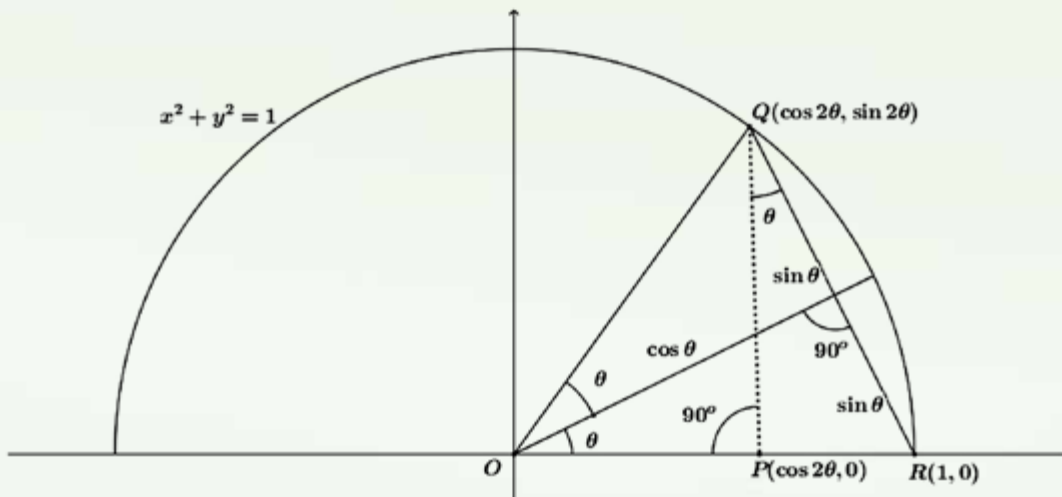
**DR M R YADAV** is working as DMS in DEO Office Narnaul. He was Formerly Expert in SCERT Gurugram (Haryana). He is a researcher in Functional Analysis. He may be contacted at [mryadavscert@gmail.com](mailto:mryadavscert@gmail.com)



**DR NARESH KUMAR** is working as an Assistant Professor in DIET Rewari. He is a researcher in Operation Research. He may be contacted at [nareshphd08@gmail.com](mailto:nareshphd08@gmail.com)

## Proof without words: Double angle formulas for sine & cosine

In this PWW, we prove without words the double angle formulas for sine and cosine.



$$(1) \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin(2\theta) = 2 \left( \frac{1}{2} \cdot 1 \cdot \cos \theta \cdot \sin \theta \right)$$

$$(2) 1 - \cos(2\theta) = \overline{PR} = \overline{QR} \cdot \sin \theta = 2 \sin^2 \theta$$

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Contributed by **Arnabi Saha** - Department of Mathematics, Jadavpur University, Kolkata, West Bengal, India  
and **Ritam Sinha** - Department of Mathematics, Ramakrishna Mission Vivekananda Centenary College,  
Rahara, West Bengal, India.

# Some more Divisibility Rules

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RAJKUMAR KANOJIYA

**D**ivisibility rules for a divisor  $p$  are algorithms that tell us whether a given number is divisible by  $p$  or not. From the lower grades we are familiar with divisibility rules for the divisors 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12, but when it comes to prime divisors greater than 10, we are clueless. In this article, I show how one can find tests of divisibility for divisors such as 13, 17, 19, 23, 29, .... I focus on divisibility by 13, 17 and 19, and give examples. These methods are not unique; readers may come up with other ways to check divisibility by these numbers.

Let  $p$  be a given divisor (e.g.,  $p = 13$ ). We start by expressing  $p$  or a multiple of  $p$  using integer multiples of its digits and addition or subtraction. We then use this relation iteratively to check divisibility. How we do this is illustrated below.

## Divisibility by 13

Let us express 13 in terms of its digits. Here is one possibility:  $13 = 1(4) + 3(3) = 13$ , so 13 can be expressed as the sum of four times its ten's digit and three times its one's digit.

Now, to check divisibility of a given number  $N$  by 13, we apply the same operation to  $N$ . That is, if  $N = 10a + b$ , we replace  $N$  by  $4a + 3b$ . Then we do this replacement operation repeatedly. At any stage, if the resulting number is a multiple of 13, we conclude that the given number  $N$  is a multiple of 13.

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*Keywords: Triangles, Ratios, Square-Roots, Exploration, Reasoning, Discussion*

### Example 1

Take the number 91. We have,  $91 \mapsto 9(4) + 1(3) = 39$ , where 39 is divisible by 13 so 91 is also divisible by 13.

### Example 2

Consider the four-digit number 1547

$1547 \mapsto 154(4) + 7(3) = 616 + 21 = 637$ . As we are not sure whether 637 is divisible by 13 or not, we repeat same procedure for 637.

$637 \mapsto 63(4) + 7(3) = 252 + 21 = 273$ . As we are not sure whether 273 is divisible by 13 or not, we repeat same procedure for 273.

$273 \mapsto 27(4) + 3(3) = 108 + 9 = 117$ . As we are not sure whether 117 is divisible by 13 or not, we repeat same procedure for 117.

$117 \mapsto 11(4) + 7(3) = 44 + 21 = 65$ . Here we know that 65 is divisible by 13.

Hence 273, 637 and 1547 are also divisible by 13.

### Example 3

Take the number 79.

$79 \mapsto 7(4) + 9(3) = 28 + 27 = 55$ , which is not divisible by 13, so 79 is also not divisible by 13.

### Divisibility by 17

Now we try to express 17 in terms of its digits. We have,  $17 = 1(3) + 7(2) = 3 + 14 = 17$ , so 17 can be expressed as the sum of three times the ten's digit and two times the one's digit.

### Example 4

Let us experiment with another number and check whether the number is divisible by 17 or not.

$34 \mapsto 3(3) + 4(2) = 9 + 8 = 17$ , as 17 is a multiple of 17, hence 34 is divisible by 17.

### Example 5

Let us try for slightly bigger numbers! Take the number 119.

$119 \mapsto 11(3) + 9(2) = 33 + 18 = 51$ , here 51 is a multiple of 17, hence 119 is divisible by 17.

### Example 6

Let us consider another number which is none other than the current year 2023.

$2023 \mapsto 202(3) + 3(2) = 606 + 6 = 612$ , here we do not know whether 612 is divisible by 17 or not, so, we repeat the procedure for 612.

$612 \mapsto 61(3) + 2(2) = 183 + 4 = 187$ , here we do not know whether 187 is divisible by 17 or not, so, we repeat the procedure for 187.

$187 \mapsto 18(3) + 7(2) = 54 + 14 = 68$ , here we know that 68 is multiple of 17, hence 187 is divisible by 17.

Therefore 612 and 2023 too are divisible by 17.

### Example 7

Let us check for 152.

$152 \mapsto 15(3) + 2(2) = 45 + 4 = 49$ , which is not divisible by 17. Hence 152 too is not divisible by 17.

### Divisibility by 19

Let us try to express 19 in terms of its digits.

$19 = 1(1) + 9(2) = 19$ , here 19 can be expressed as sum of its ten's digit and twice the one's digit.

Let us check for few numbers!

### Example 8

Take the number 57.

$57 \mapsto 5(1) + 7(2) = 5 + 14 = 19$ , as 19 is a multiple of 19, hence 57 is divisible by 19.

### Example 9

Let us try another number, 114.

$114 \mapsto 11(1) + 4(2) = 11 + 8 = 19$ , as 19 is a multiple of 19, hence 114 is divisible by 19.

### Example 10

Let us try a bigger number, 2166.

$2166 \mapsto 216(1) + 6(2) = 216 + 12 = 228$ , as we are not sure whether 228 is divisible by 19 or not, so, we repeat the procedure for 228.

$228 \mapsto 22(1) + 8(2) = 22 + 16 = 38$ , as 38 is divisible by 19, 2166 too is divisible by 19.

### Example 11

Let us try another number 2024.

$2024 \mapsto 202(1) + 4(2) = 202 + 8 = 210$ , we are not sure whether the number is divisible by 19 or not, so we repeat the procedure.

$210 \mapsto 21(1) + 0(2) = 21 + 0 = 21$ , we know that 21 is not divisible by 19 so 210 is also not divisible by 19, hence 2024 is also not divisible by 19.

Hope you all have enjoyed reading this article.



**RAJKUMAR KANOJIYA**, MSc (Mathematics), B.Ed., PGDSLML, has been working as a Mathematics Educator from the past 12 years. He works closely with students from Grades VI–X from various educational boards. He is a Microsoft Certified Educator who enjoys integrating technology and art with the teaching and learning of Mathematics. He believes mathematics is not just a subject but a language to be understood by all. He can be reached at [rajkumar.kanojiya7@outlook.com](mailto:rajkumar.kanojiya7@outlook.com)

### Closing note by the editors

Divisibility by  $k = 10x + y$  is checked by writing  $k = ux + vy$  for integers  $u, v$ . Then, to check divisibility of any given number  $n$  by  $k$ , we write  $n = 10a + b$ , compute  $au + bv$ , and we iterate this operation till we can check mentally if the resulting number is divisible by  $k$ . If it is, then  $n$  is divisible by  $k$ .

Here are some questions for exploration we pose to the reader.

1. What is the logic behind the test? How can we be sure that  $10a + b$  is divisible by  $k$  if and only if  $ap + bq$  is divisible by  $k$ ?
2. Different choices may be available for the integers  $u, v$ . How should we choose them so that the test proceeds swiftly? For example, to check the divisibility of 85 by 13 using  $(u, v) = (4, 3)$ , i.e., the test described above, the iteration proceeds very slowly:

$$85 \mapsto 32 + 15 = 47 \mapsto 16 + 21 = 37 \mapsto 12 + 21 = 33 \mapsto 21 \mapsto \dots$$

# Proof Without Words: Alternating Sum of Odd Numbers

**ARNABI SAHA**

In this visual proof, we will demonstrate that  $\sum_{k=1}^n (-1)^{n-k} (2k-1) = n$ .

**Editor's Note.** This is a visual and imaginative, though round-about way of proving this identity. We request our readers to remember that this is just a visualization, but that is the case with many such 'proofs'.

**Theorem.** We will prove that  $\sum_{k=1}^n (-1)^{n-k} (2k-1) = n$ , where  $n$  is a natural number.

**Proof.** We will provide the visualization of the theorem for  $n = 7$  and 6 respectively.

**Case-I.** First we consider the situation for odd  $n$ ; for this we show that

$$1 - 3 + 5 - 7 + 9 - 11 + 13 - \dots + (2n - 1) = n.$$

For the visualization, we take  $n = 7$ .

Let,

$$\begin{aligned} U &= 1 + 5 + 9 + 13 \\ &= 1 + (1 + (1 \times 4)) \\ &\quad + (1 + (2 \times 4)) \\ &\quad + (1 + (3 \times 4)). \end{aligned}$$

That is  $U =$

1			
1	4		
1	4	4	
1	4	4	4

So,  $2U =$

1				
1	4			
1	4	4		
1	4	4	4	

 $+$ 

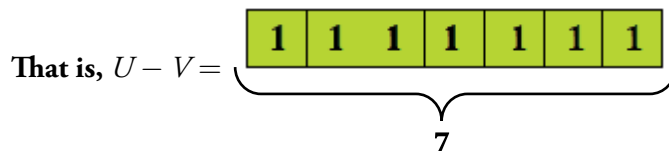
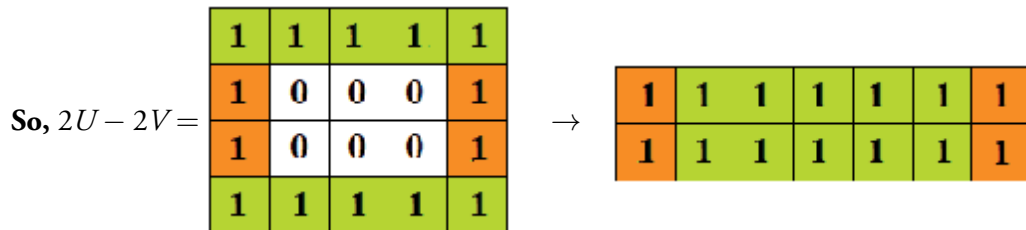
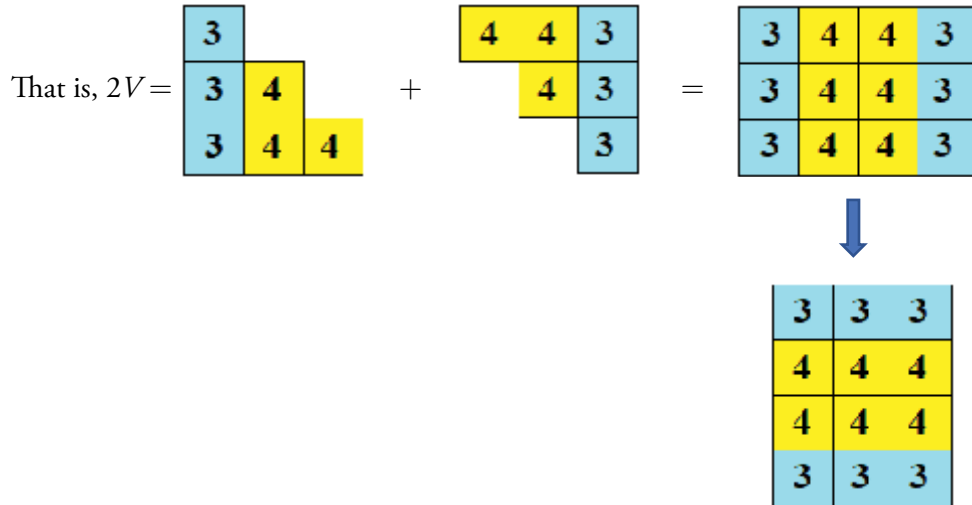
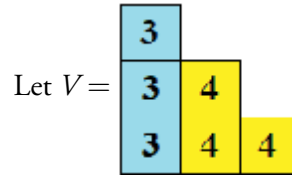
	4	4	4	1
		4	4	1
			4	1
				1

 $=$ 

1	4	4	4	1
1	4	4	4	1
1	4	4	4	1
1	4	4	4	1

Let,

$$\begin{aligned}
 V &= 3 + 7 + 11 \\
 &= 3 + (3 + (1 \times 4)) \\
 &\quad + (3 + (2 \times 4))
 \end{aligned}$$



Thus,  $1 - 3 + 5 - 7 + 9 - 11 + 13 = 7.$

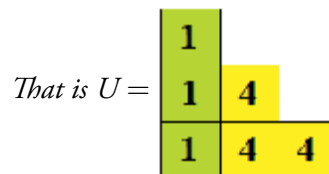
**Case-II.** Next we consider the situation for even  $n$ ; for this we show that

$$-1 + 3 - 5 + 7 - 9 + 11 - \dots + (2n - 1) = n.$$

For the visualization, we take  $n = 6.$

Let

$$\begin{aligned}
 U &= 1 + 5 + 9 \\
 &= 1 + (1 + (1 \times 4)) \\
 &\quad + (1 + (2 \times 4)).
 \end{aligned}$$



That is  $2U =$ 

1			
1	4		
1	4	4	

 $+$ 

	4	4	1
		4	1
			1

 $=$ 

1	4	4	1
1	4	4	1
1	4	4	1

Let  $V = 3 + 7 + 11$   
 $= 3 + (3 + (1 \times 4))$   
 $+ (3 + (2 \times 4)).$

That is,  $V =$ 

3		
3	4	
3	4	4

That is  $2V =$ 

3			
3	4		
3	4	4	

 $+$ 

	4	4	3
		4	3
			3

 $=$ 

3	4	4	3
3	4	4	3
3	4	4	3

That is,  $2V - 2U =$ 

2	0	0	2
2	0	0	2
2	0	0	2

So,  $V - U =$ 

1	1
1	1
1	1

 $\left. \vphantom{\begin{matrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{matrix}} \right\} 6$

Thus,  $-1 + 3 - 5 + 7 - 9 + 11 = 6.$

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**ARNABI SAHA** is a student of Master's degree at the Department of Mathematics, Jadavpur University. Arnabi's professional interests include proof and argumentation, with an emphasis on mathematical techniques that foster stronger conceptual understanding of calculus ideas. More recently, she has become interested in the visualisation of mathematics (in particular, what are known as Proofs Without Words). She may be contacted at [arnabisaha99@gmail.com](mailto:arnabisaha99@gmail.com).

# Developing an Understanding of Probability

... *Experience from the classroom*

**MANISHA VERMA &  
SANDEEP DIWAKAR**

**D**IET Bhopal and Azim Premji Foundation, Bhopal jointly organize an annual seminar on the teaching and learning of mathematics. This year, the team decided to document the teachers' work so that they may present the same in the district level seminar. In the preliminary discussions, teachers understood the processes and decided on topics and concepts that they would like to work on with the children. Manisha ji chose "probability" for this task, for the following reasons:

1. Probability has been newly included in the new NCERT Books for class 8 (Previously it was not there in the state textbooks).
2. It is interesting and related to daily life.
3. It is a topic which could engage children who are afraid of math.

She worked with 8 children enrolled in class VIII with the aim of developing the following abilities related to probability –

- Understanding the vocabulary and concept of probability and related ideas such as possible and impossible events.
- Developing the ability to evaluate all the possible outcomes for a given situation.
- Bringing a rational approach to social superstitions.
- Finding the probability of different events.

She initiated discussions with the children on a few statements such as:

- India may win the next match.
- If a circle is large, its radius is also large.

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*Keywords: Conceptual understanding, child's context, math vocabulary, rational thinking, classroom discussions, mathematical communication*

- It may rain today.
- If a dice is thrown, 6 may appear on the upper face.

The children discussed the statement “India may win the next match...” among themselves. Anikesh said, “It may or may not happen.” When other children asked the reason, he said – “it depends on the playing.” On the second statement, “the bigger the circle...” everyone seemed to agree that it is certain that if the circle is bigger then, the radius is also bigger. She had similar conversations regarding such statements related to possibility of occurrence of different events with the children. She asked some oral and written questions through which she concluded that:

- Initially only two children were able to have an idea about the probability of events.
- Only one child was able to list possible events.
- Children were not aware of the concepts/terminology related to probability; this was understandable, as they were learning these concepts for the first time.

Keeping the level of understanding of the children and the objectives of the work in mind, she planned some activities for her class -

1. Discussions and activities to understand possible and impossible events
2. Listing of outcomes
3. Introduction to related terminology through various games and activities – weighing the options/alternatives, working out possible events, understanding what equally-likely outcomes are, and relating probability with chance
4. Work on social superstitions

### 1. Discussion and activities to understand possible and impossible events

Children were asked to make statements related to possible and impossible events. Children wrote and read out some such statements:-

- a. Chances of the sun rising during the rainy season (Anikesh)

- b. Heavy rains followed by floods (Deepanshi)
- c. Rainbow sighting after rains (Jyoti)
- d. Sun rising in the East (Mohit)
- e. 32 days in the month of March (Jyoti)
- f. Madam may or may not come to school tomorrow (Saloni)
- g. Sun rising at night (Anikesh)

When a child made her/his statement, other children gave their opinion on whether it is possible or not. For statements such as “the rising of the sun during the rainy season” they agreed that it may or may not happen, so these were included in the possible events.

### 2. Listing of outcomes

Nikita asked a question to the children, “If she tries to climb the tree, what are the possibilities?” The children described different possibilities such as – you may climb a tree, you may fall from the tree, you cannot climb a tree, you can climb a tree but will not be able to descend, you can climb a tree and descend. Similarly, Dipanshi said “What are the possibilities if I want to go to Bhopal from Khurachani village to buy clothes?” The views of the children were – petrol runs out on the way, the vehicle tyre may get punctured, shop may be closed, you do not like the clothes, and you like the clothes and buy them... After a number of such discussions, we moved on to those questions which have a fixed number of possibilities.

**The examples discussed in the next part included:** If 2 green and 4 red balls are kept in a box and 1 ball is drawn then what are the possible colours of the drawn ball? What are the possible outcomes if a coin is tossed? What are the outcomes if a die is thrown? What are the outcomes if two coins are tossed simultaneously or two dice are thrown simultaneously? The children also worked with ball, coins, and dice. In this way the outcomes were worked out.

### 3. Introduction of terminology with games and activities

A race of Head and Tail - In this game, groups of two children were formed. For each group a number line was drawn on the floor in which the numbers from +5 to -5 were recorded. Each player took a counter and placed it at 0. A coin was given to each group and the rules were explained. Players will take turns tossing a coin; if heads come up, move forward one number, i.e., from 0 to +1 and if tails come up, go back one number, i.e., from 0 to -1. The player who reaches +5 or -5 first, will be the winner.

In this game, the children also recorded (i) In how many turns/rounds the game ended and (ii) how many times head or tail appeared in total. These were also discussed in large groups at the end of the game. The children concluded that the more times the coin was tossed, the closer the number of heads and of tails were.

**Editor's Note:** This documentation is a very good beginning to understand the Law of Large Numbers: empirical probability converges to theoretical probability as the number of trials increases. In this case, each round is a trial, empirical probabilities are the proportions of heads (or tails) out of all tosses, theoretical probabilities are  $P(\text{head}) = P(\text{tail}) = \frac{1}{2}$ . Since the number of heads became closer and closer to the number of tails, the empirical probabilities, i.e.,  $\frac{\text{number (no.) of tails}}{\text{no. of heads} + \text{no. of tails}}$  and  $\frac{\text{number (no.) of heads}}{\text{no. of heads} + \text{no. of tails}}$  both became closer and closer i.e., to the theoretical probabilities of these outcomes.

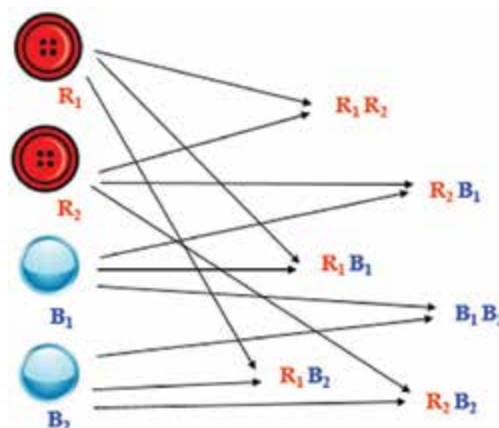
#### A game of buttons

In this game 2 red and 2 blue buttons were taken. All four buttons were put in a small envelope. Two teams of the children of the class were formed. It was decided in the game that two buttons will be drawn simultaneously (which children will take out during their turn). Teams have two options – the first is to have both buttons of the same colour (both the buttons are either red buttons or blue buttons) and second is that the buttons are of different colours (i.e.,

one red button and one blue button). The first team chose the option different colours, i.e., if one blue and one red button came out, then their team would get 1 point and if both the buttons that came out were of the same colour, then the second team would get 1 point. The team that reaches 10 points first will win the game.

The game started and the first team won the game. The children expressed their desire to play one more time. They were asked – “Which option do you want to choose – same colour buttons or different colour buttons?” Both the teams were ready to take any option. This time also the team which had opted for different colour buttons won. Then the teacher asked a question - which option is more likely to win and why? A few children said, “different buttons,” but they were not very confident and couldn't discuss the reason. The teacher encouraged them to think about the outcomes as they did in the previous activities. The children started working on the possible outcomes and weighed the options. That if one button is red, then what the options for the second button are or what kind of pairs shall be made.

The children found out that there are a total of 6 possibilities out of which 4 are in favour of those who choose different colour buttons and only 2 are in favour of those who choose the same colours. Therefore, the team that chooses the option with the different colour buttons has a double/higher chance of winning, whereas one feels that both teams have equal chances of winning. This helped a lot in understanding how



we should analyse and make careful decisions in choosing the options.

**Editor's Note:** The arrow diagram is a nice application of Cartesian Product and is a very useful tool in combinatorics to determine all possible cases. Such diagrams help children in understanding the situation much clearer compared to just textual explanation. It also enables them to draw similar diagrams to solve other problems they encounter.

### Activity with balls in a bag

5 blue and 3 pink balls were put in a bag. The children were asked if 1 ball is drawn from these, what are the chances it will be blue (or pink)? Which colour ball is more likely to be drawn? And why? Such questions helped to arrive at the different outcomes and the probability of getting each outcome.

Taking this further, questions such as, if two balls are taken out of this bag at the same time, then what can be the possible outcomes were also asked. What is the probability that both are (i) blue/pink/ of same colour or (ii) of different colours?



**Editor's Note:** Such simple activities are important to illustrate that even when there are two possibilities, i.e., in this case blue or pink, the probabilities may not be half-half. It is a common misconception that arises from equally likely outcomes (from coin tossing and dice rolling – unless the dice is suitably modified) and this is a simple way to prevent that misconception.

### Activity with cards and coins/dice to discuss related concepts

This activity consists of cards with words such as chance, outcome, impossible event, probability, equally likely outcomes, etc. Those cards were turned over and placed on the table. Along with cards, some materials like bag, balls, coins, and dice were kept. A child would come to the front and pick up a card and explain the concept written on that card in his own words with the help of the material placed on the table or on the board.

**Editor's Note:** Again, a simple idea allowing the teacher to assess the learning of the children while giving the latter the opportunity to articulate their understanding with suitable manipulatives which they are free to choose and use.

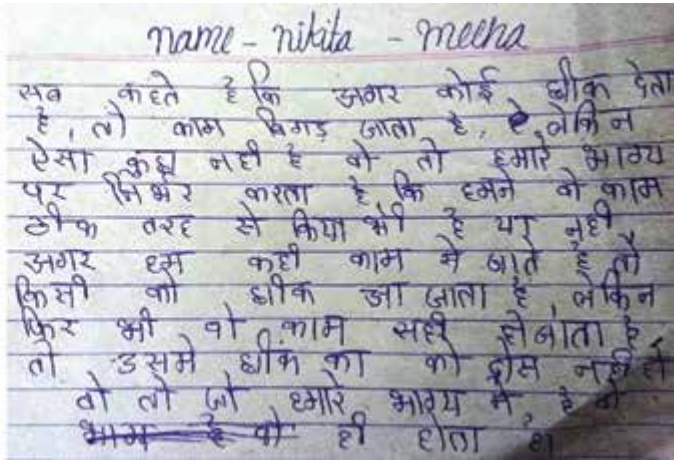
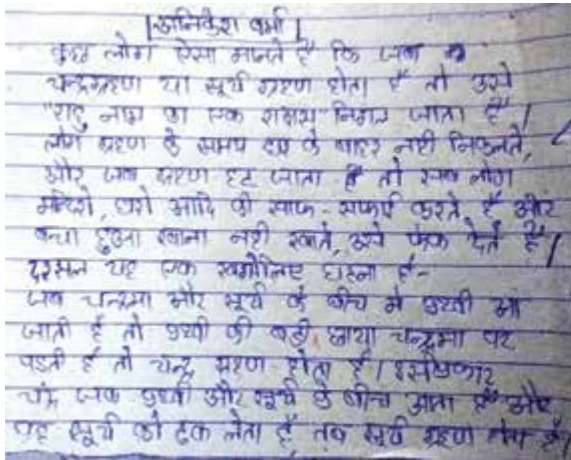
### 4. Work on social superstitions and prejudices

Children discussed their social superstitions and prejudices. Each child described the prejudice of his/her society, some of which were:

- If you see your face in the mirror as soon as you wake up in the morning, then the whole day gets spoiled.
- Money comes when the hand is itchy.
- If we sneeze, we should stop for a while and only then leave for work.
- If the crow caws near the house, then guests may be expected that day.
- If the cat crosses one's path, one should stop.

Children narrated their experiences related to each superstition and shared them in writing. Some said that this is really true, while some were not able to decide. It was decided that we should observe carefully for a few days, write them down and then discuss again. One day, their experiences were discussed in the class.

The children tried to note observations and discussed in group that allowed them to verify their beliefs, prejudices, and social superstitions. To a large extent it helped the children understand that it is not good to associate results with superstitions. E.g., most of them could



understand that sneezing has nothing to do with getting success or failure. At the same time a few of them were also associating the success / outcome with “luck”.

**Assessment**

During this whole process, we kept working on the related questions of the textbook. Manisha did oral, written and activity-based assignments again with the children for the assessment. From this assessment it emerged that –

- **Understanding of chance:** Initially only 2 out of 8 children were able to demonstrate this understanding, now 6 children understood the concept of chance and they could even explain with examples.
- **Thinking about possible and impossible events:** 5 children were able to describe possible and impossible outcomes, one more child was able to discuss after a few hints.
- **Working on the ability to make the right decisions considering all possible alternatives to an action/event:** 6 children not only described all possible outcomes, but they were also able to evaluate their choices based on this. While playing different games with buttons/counters, they were asked to choose their option(s). They explored all possible outcomes and then chose their option. For example, in a game we played with balls in a bag: 2 blue balls and 3 red balls were put in a

bag and one ball was drawn at random. The children were asked to choose the colour of the ball. They chose red, with the logic that the chance of getting a red ball is more.

- **Motivating for thinking about social prejudices:** Intense and frequent discussions with children enabled them to reason out their views on some social prejudices and superstitions.
- **Understanding the terminology/concepts related to probability:** 5 children developed a good understanding of experiment, event, chance, probability, equally likely events, etc. They could find the probabilities of different events.

**Learnings for the teacher**

- Teaching through activities not only makes it easy for the children to understand but also makes it easier for the teacher.
- Repeated discussion of various issues with the children enabled them to think and start expressing their views. They could think, express, and write their thoughts. Such opportunities helped them to assess and reflect on what we generally think.
- Children’s self-confidence increased.
- The teacher also had problem with some concepts but during this work her understanding also increased by studying and teaching.



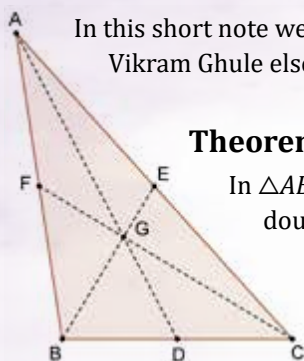
**MANISHA VERMA** has done her M.Sc (Mathematics), M.Sc (Physics) and B.Ed. She is a teacher at Govt. Middle School, Khurchani, Bhopal (M.P.) for the last 10 years. Prior to this, she has worked in Demonstration School (Regional Institute of Education, NCERT), Bhopal. She loves reading books and listening to songs. She may be contacted at [manishashrivastava309@gmail.com](mailto:manishashrivastava309@gmail.com)



**SANDEEP DIWAKAR** has been working as a mathematics resource person in Azim Premji Foundation, Bhopal, M.P. since 2012. He has experience of teaching mathematics at higher secondary school and worked in Rajya Shiksha Kendra (SCERT) Bhopal for 15 years as a lecturer. Sandeep has been associated with development of SCF, textbooks, training modules and teaching learning material for teacher educators, teachers and children. His articles have been published in Shaikshik Palash, Prathmik Shikshak, Shaikshik Sandarbh, etc. Sandeep may be contacted at [sandeep.diwakar@azimpremjifoundation.org](mailto:sandeep.diwakar@azimpremjifoundation.org)

## A THEOREM ABOUT PERPENDICULAR MEDIANS IN A TRIANGLE

*Keywords: Medians, double implication, Apollonius*



In this short note we prove a result which bears a curious similarity to the one proved by Vikram Ghule elsewhere in this issue.

### Theorem.

In  $\triangle ABC$ , let the medians be  $AD$ ,  $BE$  and  $CF$ . Then we have the following double implication:

$$BE \perp CF \iff AB^2 + AC^2 = 5BC^2.$$

**Proof:** Let  $a, b, c$  have their usual meanings. We have:

$$BE \perp CF \iff BG^2 + CG^2 = a^2 \iff 4(BE^2 + CF^2) = 9a^2, \\ \text{since } BG/BE = 2/3 = CG/CF.$$

Now, using the theorem of Apollonius:

$$2(BE^2 + CE^2) = a^2 + c^2, \quad \therefore 4BE^2 = 2a^2 + 2c^2 - b^2, \\ 2(CF^2 + AF^2) = a^2 + b^2, \quad \therefore 4CF^2 = 2a^2 + 2b^2 - c^2.$$

Therefore by addition we get:

$$4(BE^2 + CF^2) = 4a^2 + b^2 + c^2.$$

Hence  $BE \perp CF \iff 4a^2 + b^2 + c^2 = 9a^2$ , i.e.,  $BE \perp CF \iff b^2 + c^2 = 5a^2$ , as claimed.  $\square$

*Contributed by Shailesh Shirali*

# A Question about 3-Digit Numbers

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**A. RAMACHANDRAN**

Are there 3-digit numbers where the product of the first digit and the 2-digit number formed by the other digits equals the product of the last digit and the 2-digit number formed by the first two digits? In other words, if  $\overline{abc}$  is a 3-digit number, is it possible that  $a \times \overline{bc} = \overline{ab} \times c$ ?

Questions such as the above reinforce the conceptual understanding of place value, divisibility rules and symbolic notation and enable students to make a problem statement, understand constraints and reason systematically.

**Solutions are given on page 59**

Hint: The notation  $\overline{abc}$  for a 3-digit number indicates that the number is  $a \times 100 + b \times 10 + c \times 1$ .

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*Keywords: Place value, constraints, reasoning, justification*

# Math-Magic

## Generalized Fermat Numbers

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**YATHIRAJ**

Some of you might have come across the following ‘Magic’ of numbers. Take a three-digit number, say 123. Repeat the same sequence of digits to make a six-digit number (in our case 123123). Now divide this by 7. To your surprise you get an integer again, i.e., 7 completely divides 123123 (in our case we get the quotient 17589). Next, divide the quotient by 11. Again, to your surprise, the resulting number is an integer (in our case, 1599). Finally, divide the new quotient by 13. Magic! You get back 123, i.e., you have extracted the original number. Oh! Is it really magic? I mean, does it work for all three-digit numbers? Readers may stop at this point to explore whether the magic holds for other such numbers (say 516516). Not just that, readers may try to find out whether this ‘magic’ works for all six-digit numbers. (For instance, does it work for 237765?)

Now, take a four-digit number, say 1234, and repeat the sequence of digits to get an eight-digit number (12341234 in our case). Now, dividing by 7, 11 and 13 does not work. Can we now find a sequence of prime divisors such that dividing by them in turn, we recover our original number 1234? The answer is yes, and the primes are 73 and 137. That is, dividing 12341234 by 73 we get 169058. Divide this by 137. Magic! – we get back 1234. If we now want to extend this technique to five-digit sequences of numbers, what primes do we need to choose? The answer is simple and relies on the structure of the place value system.

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*Keywords: Exploration, conjecture, place value, divisibility, primes, algebra, identities, parity.*

Observe that carrying out the division by 7, 11 and 13 one after the other is the same as dividing the given number by their product that is by  $7 \times 11 \times 13 = 1001$ . Notice that, if  $\boxed{abc}$  represents a three-digit number, then

$$\boxed{abc} \times 1001 = \boxed{abc} \times (1000 + 1) = \boxed{abc000} + \boxed{abc} = \boxed{abcabc}.$$

Therefore:

$$\frac{\boxed{abcabc}}{7 \times 11 \times 13} = \boxed{abc}.$$

Thus, to see the analog of this magic for five-digit sequences of numbers, one should observe that

$$\boxed{abcdeabcde} = 100001 \times \boxed{abcde} \implies \frac{\boxed{abcdeabcde}}{100001} = \boxed{abcde}.$$

Thus, it is enough to look for the prime factorization of 100001 which is  $11 \times 9091$ . Hence the primes to be used for the divisions are 11 and 9091.

Once the trick is known, the task seems uninteresting and the thrill in the magic is lost. But the game is not over yet! We have many observations to make here. The first is that most numbers of the form  $10^n + 1$  (which has  $n - 1$  zeroes between two 1's) are composite (i.e., they have at least two prime divisors, so we can carry out the sequential division process to recover the original number). We have seen this till  $n = 5$ . It is an easy task for a computer to give us the factorization of  $10^n + 1$ , at least for small values of  $n$ . But even a computer has limitations: it cannot factorize  $10^n + 1$  beyond some  $n$ .

Then how do we know that for most natural numbers  $n$ ,  $10^n + 1$  is composite? Consider  $10^{25} + 1$ . It is easy to observe that any prime divisor of  $10^5 + 1$  is also a divisor of  $10^{25} + 1$ . How? Simple:

$$10^{25} + 1 = (10^5)^5 + 1 = (10^5 + 1)(10^{20} - 10^{15} + 10^{10} - 10^5 + 1),$$

which is based on the following identity which is true for all odd  $n$ :

$$a^n + 1 = (a + 1)(a^{n-1} - a^{n-2} + a^{n-3} - \dots + 1).$$

This gives us a general way of dealing with numbers of the form  $10^n + 1$ , where  $n$  is multiple of some odd number greater than 1. Suppose  $n = mk$ , where  $m > 1$  is odd and  $k > 1$  is any integer. Then

$$10^n + 1 = 10^{mk} + 1 = (10^k)^m + 1 = (10^k + 1) \left( (10^k)^{m-1} - (10^k)^{m-2} + \dots + 1 \right).$$

Obviously, a factor of  $10^k + 1$  is also a factor of  $10^n + 1$ . We deduce that  $10^n + 1$  is composite if  $n$  is a multiple of an odd number greater than 1.

**Challenge for the reader.** Suppose  $n = 10^9 + 1 = 1000000001$ . Can you find at least three prime divisors of  $n$  without the help of a computer?

However, what happens if  $n$  is not a multiple of any odd number greater than 1? For example,  $n = 4$  is such a number. We know that any natural number greater than 1 can be written as a product of primes in a unique manner; this is the *fundamental theorem of arithmetic*. If  $n$  is not divisible by any odd number, then  $n$  must not be divisible by any odd prime. Hence the prime factorization of  $n$  must contain only even primes. The only even prime is 2. It follows that  $n$  is of the form  $2^k$ . Since  $n > 2$ , we must have  $k > 1$ . Can we now say that in such a case  $10^n + 1$  is composite? The answer is: 'we do not know.'

At present, we do not know the primality status of all the numbers  $10^{2^k} + 1$  for natural numbers  $k$ ; there are both prime numbers and composite numbers in the set of all such numbers. In exploring this question, we are led into the serious business of number theory!

Explorations of this kind were started with Pierre De Fermat when he wrote a letter to his friend Frenicle in 1640. Fermat expressed his belief that numbers of the form  $F_n = 2^{2^n} + 1$  are primes for all integers  $n \geq 0$ . Although he had no proof, he believed it was true. There was a reason for Fermat to believe in it as for the first few values of  $n$ , that is for  $n = 0, 1, 2, 3$  and  $4$ ,

$$F_0 = 3; F_1 = 5; F_2 = 17; F_3 = 257; F_4 = 65537,$$

which are all primes. If it were true that  $F_n$  is prime for all non-negative integers  $n$ , we then would have a sequence generating only primes. This belief of Fermat was supported by another great mathematician, Mersenne. But when this problem was brought to the notice of Euler by Goldbach, Euler was able to show that  $F_5$  is a composite number with 641 as one of its prime factors.

**Challenge.** Suppose that (like Fermat) we had no calculator or electronic computer. How could we check whether 65537 is prime or composite?

Do you know how big is  $F_5$ ? It is 4294967297. Oh! Did Euler divide  $F_5$  by every prime number less than or equal to the square root of  $F_5$  to find its factors? That sounds extremely unlikely! Then how could he have shown that 4294967297 is composite? We shall not describe in detail the method of Euler. We shall simply give a road map. Interested readers may go through Chapter 10 of the book *Journey through genius* by William Dunham for a detailed proof by Euler.

### Outline of Euler's proof

Euler proved the following statement which can be generalized easily.

**Euler's statement.** If  $a$  is an even number and  $p$  is a prime number that divides  $a^{32} + 1$ , then  $p$  must be of the form  $64k + 1$  for some  $k$ .

**Generalization.** If  $a$  is an even number and  $p$  is a prime number that divides  $a^{2^n} + 1$ , then  $p$  must be of the form  $2^{n+1}k + 1$  for some positive integer  $k$ .

In the above, replacing  $a$  by 2, Euler listed the possible candidates for the prime factors of  $2^{2^5} + 1$ ; they are 65, 129, 193, 257, 321, 385, 449, 513, 577, 641, .... From this we eliminate candidates that are themselves not prime; we are left with the list 193; 257; 449; 577; 641; ... Using the familiar long division method, Euler found that 641 divides  $2^{2^5} + 1$  and thus showed that

$$2^{2^5} + 1 = 641 \times 6700417.$$

### On numbers of the form $a^{2^n} + 1$

Numbers of the form  $2^{2^n} + 1$  are called *Fermat numbers*, and numbers of the form  $a^{2^n} + 1$  are called *generalized Fermat numbers*. If they are primes, then they are called *Fermat primes* or *generalized Fermat primes* (respectively). Here are some interesting facts about these numbers.

1. A prime factor of  $a^{2^n} + 1$  is of the form  $k2^m + 1$  for some integers  $k$  and  $m > n$ .
2. Every prime of the form  $k2^m + 1$  (with  $m > 1$ ), is a factor of  $a^{2^n} + 1$  for at least one even  $a$ .

3. The largest known generalized Fermat prime is the number

$$1963736^{2^{20}} + 1.$$

See <https://t5k.org/primes/page.php?id=134423>.

4. Fermat had a connection with the Greeks. The connection is seen in the topic of geometric constructions. The Greeks were keenly interested in constructing regular polygons using only the compass and ruler. (Here, the term ‘ruler’ refers to a straight edge with no markings on it.) They knew how to construct a regular triangle (i.e., an equilateral triangle), a square, a regular pentagon, a regular hexagon, etc. They wondered: Is it possible to construct all such regular polygons using compass and ruler? They did not have an answer to this question.

It took almost 2000 years before a complete answer to this question became known, through the work of Gauss and Wantzel. It turns out that the answer is ‘No; only some regular polygons can be so constructed.’ Their famous result is that the only regular polygons which can be constructed using compass and ruler are those with number of sides of the form  $2^k p_1 p_2 \dots p_n$ , where  $k \geq 0$ ,  $n \geq 1$ , and  $p_1, p_2, \dots, p_n$  are unequal Fermat primes. Hence, it is not possible to construct a regular polygon with 7 sides or 9 sides or 11 sides or 13 sides or 19 sides; but it is possible to construct a regular polygon with 15 sides or 17 sides! What an astonishing conclusion to a 2000-year-old problem!



**YATHIRAJ** currently works in the department of Mathematics, Sarada Vilas College, Mysuru. Prior to this, he worked in Azim Premji Foundation, Mandya as a Teacher Educator for nearly four years. He is doing research in the areas of special functions and number theory. He is also interested in understanding the patterns, connections and thought processes which one employs during problem solving. He may be contacted at [yathirajsharma@gmail.com](mailto:yathirajsharma@gmail.com).

# A Conjecture equivalent to the Goldbach Conjecture, and some Consequences

K SASIKUMAR

The *Goldbach conjecture* is one of the oldest and best-known unsolved problems in Number Theory. It states the following:

**Conjecture 1** (Goldbach). *Every even natural number greater than 2 can be written as the sum of two prime numbers.*

For example:  $6 = 3 + 3$ ,  $8 = 5 + 3$ ,  $10 = 5 + 5$ , ...,  $2022 = 1009 + 1013$ , ...

It was conjectured in the year 1742 by Christian Goldbach.

Closely related to this conjecture is the *Odd Goldbach Conjecture* (or the 3-primes problem):

**Conjecture 2** (Odd Goldbach Conjecture). *Every odd integer greater than 5 can be written as the sum of 3 primes.*

For example:  $7 = 2 + 2 + 3$ ,  $9 = 3 + 3 + 3$ ,  $11 = 3 + 3 + 5$ , ...

It is easy to see that the Goldbach Conjecture implies the Odd Goldbach Conjecture.

*Keywords: Goldbach conjecture, Weak Goldbach Conjecture, Odd Goldbach Conjecture, Maillet Conjecture, prime number*

**Proof.** Consider an odd integer  $n \geq 7$ . We must show that it can be written as the sum of 3 primes. Consider the even number  $n - 3$ . Since  $n - 3 \geq 4$ , the Goldbach Conjecture (assumed to be true) can be applied. Hence there exist primes  $p$  and  $q$  such that  $n - 3 = p + q$ . This yields  $n = 3 + p + q$ . We have thus written  $n$  as a sum of 3 primes.  $\square$

Since the Goldbach Conjecture implies the Odd Goldbach Conjecture, we refer to the Odd Goldbach Conjecture as the *Weak Goldbach Conjecture*.

Here is another easy consequence of the Goldbach Conjecture:

**Conjecture 3.** *Any square number  $n > 4$  can be written as the sum of 3 prime numbers.*

For example:  $16 = 2 + 7 + 7$ ,  $36 = 2 + 11 + 23$ , ...

**Proof.** Consider a square number  $n$ . We subdivide the proof into two parts.

Case (i)  $n$  is an odd square number greater than 4. In this case, by the previous claim  $n$  can be written as the sum of 3 primes.

Case (ii)  $n$  is an even square number greater than 4. Let  $n = 4m^2$  where  $m$  is a positive integer greater than 1. Then  $n - 2 = 4m^2 - 2$  is even and greater than or equal to 4. Therefore, by the Goldbach Conjecture, there exist primes  $p_1$  and  $p_2$  such that  $n - 2 = p_1 + p_2$  or  $n = 2 + p_1 + p_2$ . We have thus written  $n$  as the sum of 3 primes.

We conclude that if the Goldbach conjecture is true, then any square number greater than 4 can be written as the sum of 3 primes.  $\square$

We now offer a new conjecture which is equivalent to the Goldbach conjecture.

**Conjecture 4 (Golden Conjecture; Sasikumar K).** *For any natural number  $n > 1$ , there exist prime numbers  $p$  and  $q$  such that  $n^2 - pq$  is a square.*

For example: (a) For  $n = 4$ , we may take  $(p, q) = (3, 5)$ . (b) For  $n = 6$ , we may take  $(p, q) = (5, 7)$ . (c) For  $n = 9$ , we may take  $(p, q) = (5, 13)$ .

**Theorem 1.** *The Golden Conjecture and the Goldbach Conjecture are equivalent to each other.*

**Proof.** We shall first show that Goldbach Conjecture  $\implies$  Golden Conjecture. So let us suppose that the Goldbach Conjecture is true. Let  $n$  be a positive integer greater than 1. Then there exist prime numbers  $p$  and  $q$  such that  $2n = p + q$ .

Now consider the quadratic equation  $x^2 - 2nx + pq = 0$ , i.e.,

$$x^2 - (p + q)x + pq = 0.$$

The roots of this equation are clearly the integers  $p$  and  $q$ . As the roots are integers, the discriminant must be a square, which means that  $4n^2 - 4pq$  is a square. Hence  $n^2 - pq$  is a square, which proves the Golden Conjecture.

Next we must show that Golden Conjecture  $\implies$  Goldbach Conjecture. So let us suppose that the Golden Conjecture is true. Let  $m \geq 4$  be an even integer. Then  $m = 2n$  where  $n > 1$  is an integer.

By the Golden Conjecture, there exist prime numbers  $p$  and  $q$  (with, say,  $p \geq q$ ) such that  $n^2 - pq$  is a square, say  $n^2 - pq = k^2$ . We naturally have  $0 \leq k < n$ .

Consider the quadratic equation  $x^2 - 2nx + pq = 0$ . Its discriminant is  $4n^2 - 4pq = 4k^2$  which is a square, so the equation has integer solutions  $n + k$  and  $n - k$ . Let

$$u = n + k = n + \sqrt{n^2 - pq}, \quad v = n - k = n - \sqrt{n^2 - pq}.$$

Note that  $u + v = 2n$  and  $uv = pq$ . Also,  $u > 1$  (since  $n^2 - pq \geq 0$ ).

We shall show that  $v > 1$  too. Since  $v > 0$ , we have  $v \geq 1$ . So it is sufficient if we prove that  $v \neq 1$ .

Suppose that  $v = 1$ . Then  $u = pq$ , hence  $pq + 1 = 2n$ , so  $n = \frac{1}{2}(pq + 1)$ . Therefore

$$\begin{aligned} \left(\frac{pq + 1}{2}\right)^2 - pq &= k^2, \\ \therefore (pq - 1)^2 &= 4k^2, \\ \therefore (pq + 2k - 1)(pq - 2k - 1) &= 0. \end{aligned}$$

Therefore either  $pq = -2k + 1$  or  $pq = 2k + 1$ . Both the possibilities imply that  $pq$  is odd. But this contradicts the statement made earlier that  $pq + 1 = 2n$ , an even number.

We conclude that the Golden Conjecture implies the Goldbach Conjecture. Therefore, the Golden Conjecture and the Goldbach Conjecture are equivalent to one another. □

**Conjecture 5** (Maillet). *Every even positive integer can be expressed as the difference of two primes.*

For example:  $2 = 5 - 3$ ,  $6 = 11 - 5$ ,  $8 = 11 - 3$ ,  $10 = 13 - 3$ ,  $12 = 17 - 5$ , ...

A very much stronger form of the above conjecture is the following.

**Conjecture 6** (de Polignac). *Every even number can be expressed as the difference of two consecutive primes in infinitely many ways.*

For example:  $4 = 11 - 7 = 41 - 37 = 83 - 79 = 101 - 97 = \dots$ . Conjecture 6 obviously implies Conjecture 5.

It is remarkable that all these conjectures continue to remain open despite enormous research efforts to prove or disprove them!

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**SASIKUMAR K** is presently working as a PG Teacher in Mathematics at Jawahar Navodaya Vidyalaya, North Goa. He completed his M Phil under the guidance of Dr.K.S.S. Nambooripad. Earlier, he worked as a maths Olympiad trainer for students of Navodaya Vidyalaya Samiti in Hyderabad. He has research interests in Real Analysis and Commutative Algebra. He may be contacted at [112358.ganitham@gmail.com](mailto:112358.ganitham@gmail.com).

# Solution to a Question about 3-Digit Numbers

## Problem may be found on page 51

There are many ways of finding the solution, one of these is outlined below.

From the question (and the hint) given on page .... it should be easy to form an equation reflecting the requirement  $a \times \overline{bc} = \overline{ab} \times c$ .

Simplify this to the extent possible. Now, one must find values of  $a, b, c$  that satisfy this equation, from the set 1 to 9. Values 1 to 9 can be assigned to  $a$  and  $b$  in turn, and the corresponding value of  $c$  computed. Cases where  $c$  belongs to the set 1 to 9 are solutions.

Ignore trivial solutions like 111, 222, etc. Tables like the one below, with values of  $a$  along the rows and values of  $b$  along the columns can help you. A sample entry is given. It gives the trivial solution 111.

	Values of $b$								
$10ab$	1	2	3	4	5	6	7	8	9
1	10								
2									
3									
4									
5									
6									
7									
8									
9									

	Values of $b$								
$9a + b$	1	2	3	4	5	6	7	8	9
1	10								
2									
3									
4									
5									
6									
7									
8									
9									

Values of  $a$

There are only four solutions: 164, 195, 265, 498.



**A. RAMACHANDRAN** has had a longstanding interest in the teaching of mathematics and science. He studied physical science and mathematics at the undergraduate level and shifted to life science at the postgraduate level. He taught science, mathematics, and geography to middle school students at Rishi Valley School for two decades. His other interests include the English language and Indian music. He may be contacted at [archandran.53@gmail.com](mailto:archandran.53@gmail.com).

# Guess the Card

SHREYA MUNDHADA

One Saturday, during our weekly class of the All-Girls Math Nurture Camp conducted by *Raising A Mathematician Foundation*, our teacher Mr. Vinay Nair gave us an interesting problem based on a card trick.

A magician and his assistant perform the trick. Someone from the audience picks at random any five cards from a complete deck of playing cards (no Jokers). The remaining cards are kept aside. The selected five cards are handed over to the assistant, without the magician seeing which cards have been chosen. From these five cards, the assistant selects one card and gives the remaining four cards to the magician, one by one. The assistant decides the order in which to give the four cards. By looking at the four cards, the magician works out the identity of the fifth card. There is no sign language for communication between the magician and the assistant; the only communication between them lies in the way the cards are handed over. The magician finds the unknown card simply by looking at the four cards.

Swasti, Tara, Akshita, Srividya and I took up the challenge to find how it could be done.

Swasti guessed that the first of the four cards handed to the magician would indicate the suit of the unknown card. There must be at least one suit with at least two cards in a set of five cards. This can be proven by the Pigeonhole Principle, as there are 5 cards ('pigeons') and 4 suits ('pigeonholes'). The suit now being known, the number on the card remains to be found.

Tara suggested that the number may be found from the way the cards were given to the magician by the assistant — face up (FU) or face down (FD). This is a pre-decided code between the assistant and the magician. Continuing with Swasti's idea, the assistant and magician decide that the suit of the unknown card must be the suit of the first card handed over to the magician. Thus, when the magician receives the first card, he knows that

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*Keywords: magic and mathematics, card trick, reasoning, making connections, permutations*

the unknown card is among the remaining twelve cards in the suit. According to their code, if the assistant hands over the first card as face up, the unknown card must be one of the first 6 numbers and if it is face down, it must be one amongst the last 6. If the second card is handed over as face up, the unknown card must be in the first 3 cards of the suit; if it is face down, the unknown card must be among the last 3 in the suit. If the third card is face up, it must be the centre card and if it is face down, it must be one of the remaining 2 cards. If it is among the remaining 2 cards, it can be determined by the fourth card. If it is face up, it is the lower card and if it is face down, it is the higher card. For instance, if the suit is spade and the first card is 3 and the unknown card is 7, then the assistant would give the first card as face up indicating that the unknown card is amongst Ace, 2, 4, 5, 6 or 7. Since the unknown card is amongst the last three in this set (5, 6 and 7), the assistant will give the second card as face down to indicate this. As the unknown card is one of the two remaining cards, that is, 5 and 7, the assistant will give the third card face down. The assistant must give the fourth card face down to indicate that the unknown card is the higher of these, that is, the 7 of Spades. Voila! Mathemagic!

Akshita suggested that one might use the spacing between the cards when they are placed on the table while handing them over to the magician, rather than handing over the cards face up or down. This is more subtle than placing cards face up or face down. However, the magician and assistant must take precautions to ensure that they can clearly distinguish the spacing.

Srividya came up with a solution by which the card could be found by looking at just the first 3 cards. Furthermore, it makes use of four ways to give a card, i.e., Vertical Face Up (VFU), Vertical Face Down (VFD), Horizontal Face Up (HFU) and Horizontal Face Down (HFD). Horizontal and vertical refer to the direction of the orientation of the card.

The magician knows that the unknown card is from the same suit as the first card. Thus, after seeing the first card, there are 12 numbers left

for the unknown card. We rank those numbers in ascending order as 1 to 12. If the first card is given as HFU, it means the unknown card is the first amongst those ranks. If the first card is given as HFD, then it means that the unknown card is the second amongst those 12 ranks, and so on as follows:

- First Card:
  - HFU = 1<sup>st</sup>
  - HFD = 2<sup>nd</sup>
  - VFU = 3<sup>rd</sup>
  - VFD = 5<sup>th</sup> to 12<sup>th</sup>
- First and second card together = 4<sup>th</sup>
- Second Card:
  - HFU = 5<sup>th</sup>
  - HFD = 6<sup>th</sup>
  - VFU = 7<sup>th</sup>
  - VFD = 9<sup>th</sup> to 12<sup>th</sup>
- Second and third card together = 8<sup>th</sup>
- Third Card:
  - HFU = 9<sup>th</sup>
  - HFD = 10<sup>th</sup>
  - VFU = 11<sup>th</sup>
  - VFD = 12<sup>th</sup>

The rules in the above procedure require the magician and the assistant to remember quite a few things, but it helps the magician guess the unknown card with just three cards.

I presented a solution which, like Srividya's, uses four ways to give a card. The first card, as in all other solutions, shows the suit of the unknown card. The value is determined as shown below:

- First Card:
  - HFU = Ace
  - HFD = 2
  - VFU = 3
  - VFD = 4 and above;
- Second Card:
  - HFU = 4

HFD = 5  
VFU = 6  
VFD = 7 and above;

- Third Card:

HFU = 7  
HFD = 8  
VFU = 9  
VFD = 10 and above;

- Fourth Card:

HFU = 10  
HFD = Jack  
VFU = Queen  
VFD = King

Whenever a card is determined, all cards received after that are irrelevant.

However, none of these were what Vinay Sir had in mind. His solution was different, he said, and the session ended with a tantalizing hint that his solution was related to Permutations.

After the class, I sent the following solution to him and Vinay Sir confirmed that this was what he had in mind.

The first card shows the suit of the unknown card. Also, if the first card is face up, the number is one of the first 6 and if it is face down, it is among the last 6. From the remaining three cards, let the highest be  $a$ , the second be  $b$ , and the last be  $c$ , where ace is the smallest and king is the highest. If two cards have the same number, the order of suits from lowest to greatest is Hearts, Clubs, Diamonds, and Spades. The three cards may be arranged in 6 distinct orders viz.  $abc$ ,  $acb$ ,  $bac$ ,  $bca$ ,  $cab$ , and  $cba$ . Each order indicates the 1<sup>st</sup>, 2<sup>nd</sup>, 3<sup>rd</sup>, 4<sup>th</sup>, 5<sup>th</sup>, and 6<sup>th</sup> numbers respectively.

So, we arrived at 4 distinct solutions for how the magician could have guessed the unknown card based on the way the assistant arranged the remaining cards.

On further thinking, I realized that it is possible to do the same trick by picking four cards instead of five where only three cards indicate the fourth (unknown) card.

The first card indicates the suit. However, since there are only four cards, all four may possibly be of different suits. Therefore, the color of the first card is the same as that of the unknown card. If the first card is face up, it is of the same suit. If it is face down, it is of the other suit (of the same color).

The second card and third card indicate the number on the unknown card.

Each card can be placed in 4 ways — Horizontal Face Up (HFU), Horizontal Face Down (HFD), Vertical Face Up (VFU) and Vertical Face Down (VFD).

The second card indicates the number on the card mod 4. Let HFU, HFD, VFU and VFD stand for 0, 1, 2, and 3 respectively. For the third card, let HFU, HFD, VFU and VFD stand for  $x$ ,  $x + 4$ ,  $x + 8$  and  $x + 12$  respectively, where  $x$  is the value derived by the second card. Thus, if the second card indicates 2 and the third card indicates  $x + 4$ , the number on the unknown card is 6. The suit is decided by the first card.

We have therefore established that one does not require 4 cards to guess the unknown card. It can be done with 3.

Can it be done with 2 cards?

I don't know... yet.

I'm working on it. If you get it before me, let me know!



**SHREYA MUNDHADA** is a homeschooled 13-year-old born in Mumbai, raised in a rural area in central India and currently living in Bangalore. She is passionate about mathematics and physics. She considers it a perfect start to the day to be woken up by her pet dog Cookie. She can be reached at [shreyamundhada1@gmail.com](mailto:shreyamundhada1@gmail.com).

# Counting Perimeter Magic Triangles

JAYADITYA GUPTA

## The problem

Arrange numbers from 1 to 9 in the circles (Figure 1) such that the sum of numbers on each side is the same, without repeating any of the numbers (so each number appears exactly once). Such a configuration is called a perimeter magic triangle of order 4 (or simply a magic triangle; we leave out the word 'perimeter'). The number of circles on each side determines the 'order' of the magic triangle. Is such a configuration possible? If so, how many solutions exist?

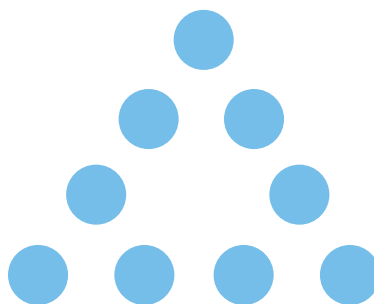


Figure 1. Perimeter magic triangle of order 4

## Generalised problem

Construct magic triangles of orders 5, 6, and 7, using the numbers from 1 to 12, 1 to 15, and 1 to 18 (respectively), by increasing the number of circles by 1 on each side in Figure 1. How many solutions are there?

If the number of circles on each side is 4 (as in Figure 1), then the numbers to be used will be from 1 to  $3(4 - 1) = 9$ . If the number of circles on each side is  $n$ , then the numbers to be used will be from 1 to  $3(n - 1)$ .

*Keywords: Magic triangle, magic sum, partitions, counting, reasoning*

In this article, I study the problem of counting the number of perimeter magic triangles of order  $n$ . The numbers used are the following:  $1, 2, 3, \dots, 3(n-1)$ . The condition is that the sums of the numbers on the three sides must be equal. It should be clear that manually counting the triangles is very tedious; we require a systematic approach.

### Important terms and notation

Below are the definitions of terms that are going to be used to solve the problem.

- **Order:** The number of circles on each side of the Magic Triangle.
- **Magic Sum (S):** The sum of the numbers on every side of the triangle is the Magic Sum of the triangle, denoted by  $S$ .
- **Partitions:** A partition of  $n$  is an expression of  $n$  written as a sum of positive integers. The order in which the integers occur is not important; the sum is an unordered sum. For example,  $4 + 2 + 1$  and  $3 + 2 + 2$  are partitions of 7. We are particularly interested in partitions where the summands are *distinct* (i.e., numbers are not repeated). The partitions are then referred to as *distinct partitions*.
- **Middle Numbers:** All the numbers excluding those at the vertices:  $p, q, r, s, t, u$ .

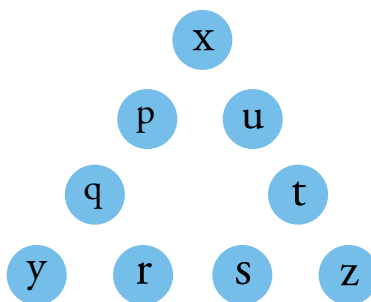


Figure 2

- **Middle Sum:** The sums of all the middle numbers on each side:  $p + q, r + s, t + u$ .

### A method to solve the problem for order 4

**Approach 1:** (To find the middle numbers in Figure 2)

**Step 1:** For the magic sum  $S$  and vertices  $(V_1, V_2, V_3)$ , where  $V_1 < V_2 < V_3$ , the middle sums are  $S_1 = S - (V_1 + V_2)$ ,  $S_2 = S - (V_2 + V_3)$  and  $S_3 = S - (V_1 + V_3)$ .

**Step 2:** We need to find the number of ways of partitioning  $(S_1, S_2, S_3)$  using distinct middle numbers.

Let the partitions be  $S_1 = M_1 + M_2$ ,  $S_2 = M_3 + M_4$ , and  $S_3 = M_5 + M_6$ , where the middle numbers and vertex numbers are distinct.

**Step 3:** Enter each number into the Magic Triangle (Figure 2).

**Approach 2:** (Finding the bounds on the magic sum)

**Step 1:** To find the smallest magic sum, the three smallest numbers, i.e., 1, 2 and 3 must be at the vertices. In finding the sum of all the numbers on all the sides, each vertex would be counted twice.

The sum of the numbers on all the sides would be (the sum of the vertices + the sum of all the numbers).

Therefore, the magic sum on each side would be  $\frac{\text{the sum of the vertices} + \text{the sum of all the numbers}}{3}$ .

**Step 2:**

$$\begin{aligned} \text{Smallest Magic Sum} &= \frac{(\text{sum of the three smallest numbers} + \text{the sum of all the numbers})}{3} \\ &= \frac{(1 + 2 + 3 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9)}{3} = 17. \end{aligned}$$

**Step 3:**

$$\begin{aligned} \text{Largest Magic Sum} &= \frac{(\text{sum of the three largest numbers} + \text{the sum of all the numbers})}{3} \\ &= \frac{(7 + 8 + 9 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9)}{3} = 23. \end{aligned}$$

So, for magic triangle of order 4, the magic sum  $S$  must satisfy the condition  $17 \leq S \leq 23$ .

Similarly, for a Magic Triangle of Order  $N$ , the range of the sums on each side is:

$$\begin{aligned} &\frac{1}{3} \left( 6 + \frac{(3n-3)(3n-2)}{2} \right) \text{ to } \frac{1}{3} \left( 9n - 12 + \frac{(3n-3)(3n-2)}{2} \right), \\ \text{i.e., } &2 + \frac{(n-1)(3n-2)}{2} \text{ to } 3n - 4 + \frac{(n-1)(3n-2)}{2}. \end{aligned}$$

Now that we have found bounds on the magic sum, we can count the number of magic triangles for each magic sum.

### An example of Approach 1

We try to form a Magic Triangle with vertex numbers (1, 2, 3) and magic sum 17.

**Step #1:** The Middle Sums are  $17 - (1 + 2) = 14$ ;  $17 - (2 + 3) = 12$ ; and  $17 - (1 + 3) = 13$ .

**Step #2:** We must find distinct partitions of (12, 13, 14) using the numbers (4, 5, 6, 7, 8, 9).

Partitions of 12	Partitions of 13	Partitions of 14
8 + 4	9 + 4	9 + 5
7 + 5	8 + 5	8 + 6
—	7 + 6	—
Total: 2	Total: 3	Total: 2

**Step #3:** We form pairs, such that the digits do not repeat as repetition is not allowed.

Pairs in Set 1: (8 + 4), (7 + 6), (9 + 5) | Pairs in Set 2: (7 + 5), (9 + 4), (6 + 8).

The resulting magic triangles are shown in Figures 3 and 4.

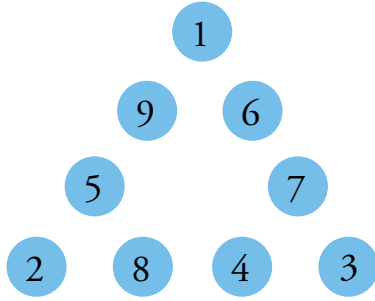


Figure 3. Magic triangle 1

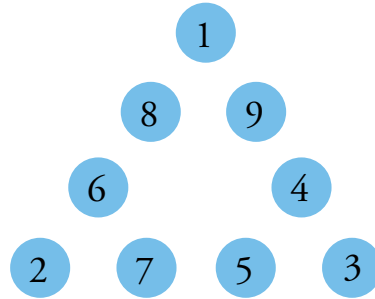


Figure 4. Magic triangle 2

### Complement

The lowest number that one can use is 1, while the highest number is 9. Note that  $1 + 9 = 10$ .

Thus, on taking the complement of 10 from each number, i.e., on subtracting each number in the magic triangle from 10, another magic triangle is formed. In this way, one can generate another magic triangle from an existing magic triangle.

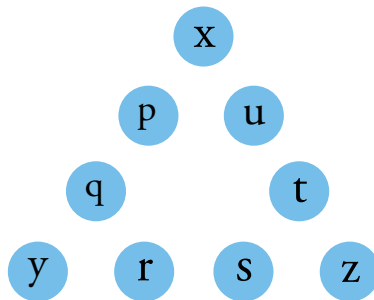


Figure 5. Using complements to generate another magic triangle

In Figure 5,  $x + p + q + y = y + r + s + z = z + t + u + x$ , so the quantities  $(10 - x) + (10 - p) + (10 - q) + (10 - y)$ ,  $(10 - y) + (10 - r) + (10 - s) + (10 - z)$  and  $(10 - z) + (10 - t) + (10 - u) + (10 - x)$  are all equal.

If the sum of a magic triangle of order 4 is  $S$ , then the magic sum of the new magic triangle would be  $40 - S$ .

For a magic triangle of order  $n$ , the lowest number is 1 and the highest number is  $3(n - 1)$ . Since  $3(n - 1) + 1 = 3n - 2$ , on taking the complement of each number from  $3n - 2$ , another magic triangle is obtained. So, we obtain the same number of magic triangles with magic sum  $S$  and  $3n - 2 - S$ .

### The algorithm (for order 4)

Program(In Java) + Output: [Generating Magic Triangles - Order 4 \(Code + Output\)](#)

The first part of the program generates partitions of the middle sum, with a given sum and given vertex numbers. A loop runs which determines the range of the possible sum and the vertices. This is then stored in a list. Finally, all the partitions are then retrieved from the list and a triplet is generated such that no number repeats and the solution is printed.

### A generalised algorithm for order n

For a generalized algorithm of order n, we would have to change the number of loops for generating the partitions as the partition size depends on n. The partition size would be (n-2). Additionally, we would have to change the number of variables to retrieve all the values.

Order 5 : Program:- [Generating Magic Triangles - Order 5 \(Code\)](#)

Order 6 : Program:- [Generating Magic Triangles - Order 6 \(Code\)](#)

Order 7 : Program:- [Generating Magic Triangles - Order 7\(Code\)](#)

### Reference values for magic triangles

Order	3	4	5	6	7	8	9	10
Number of Magic Triangles	4	18	700	13,123	316,424	7,317,145	176,476,738	4,279,366,371

### A very special magic triangle

See Figures 6 and 7. They show a magic triangle of order 4 and magic sum 20, and another triangle with the squares of all the numbers in the original triangle. Amazingly, the second triangle is also a magic triangle (with magic sum 126). This is truly a remarkable occurrence.

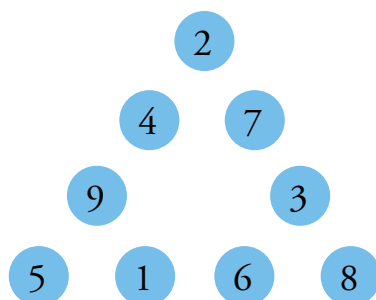


Figure 6. Original magic triangle

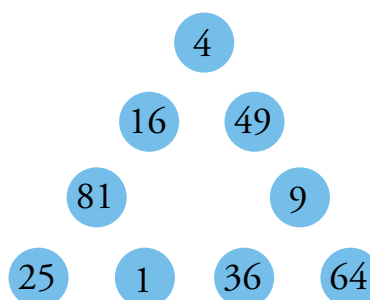


Figure 7. The squared magic triangle

Similarly, there exist magic triangles with different orders which when squared yield another magic triangle. There is 1 such triangle for order 4; 4 such triangles for order 7; and 9 such triangles for order 8. In the case of orders 5 and 6, there are no such triangles.

### References

1. [Theorem Concerning A Magic Triangle](#)
2. [On Adding Magic Triangles](#)
3. [Perimeter Magic Triangles](#)



**JAYADITYA GUPTA** is a 9th grade student at Dhirubhai Ambani International school. He is a math enthusiast who is passionate about learning higher maths concepts and finding its applications in daily activities. His favourite fields include Number Theory and Combinatorics. He also loves to leverage technology for complex mathematical solutions, play the tabla and find relationships between music and mathematics. He may be contacted at [jayaditya.gupta1903@gmail.com](mailto:jayaditya.gupta1903@gmail.com)

# Exploring Geometrical Constructions – The GeoGebra Way

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JONAKI GHOSH

Performing constructions of geometrical figures using a compass and straight edge (that is, a ruler without markings) are an integral part of learning mathematics at school. These are typically introduced in the middle school and students generally enjoy this topic as it gives them an opportunity to work with their hands. Teachers spend a significant amount of time teaching the “steps of construction” which students follow as an algorithm to arrive at the correct constructed figure. However, this approach does not leave much scope for exploration of figures and their geometrical properties. In this article, we shall illustrate the opportunities provided by GeoGebra, a Dynamic Geometry Software (DGS), in exploring constructions of simple geometrical figures through multiple approaches.

The reader may wish to read the introductory article *Dynamic Geometry Software: A Conjecture Making Tool* which appeared in the Techspace section of the July 2020 issue, in order to recall some basic features of GeoGebra.

## **Constructing an equilateral triangle**

An equilateral triangle is one of the easiest figures to construct. Usually one of two methods, using compass and ruler, are taught to students.

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*Keywords: Geometric constructions, Dynamic Geometry Software (DGS), exploration, experimentation, drag test, properties of polygons.*

*Method 1:* Draw a straight line segment AB assumed to be the base of the required triangle. Place the compass point on vertex A, stretch the pencil point of the compass to vertex B and draw an arc somewhere above the line segment AB. Now place the compass point on vertex B and using the same length draw a new arc, which cuts the previous arc. The point of intersection of these two arcs is the third vertex (say C) of the equilateral triangle. Once vertex C is joined to A and B we have an equilateral triangle. See Figure 1(a).

*Method 2:* Draw a straight line segment AB, which is assumed to be the base of the required triangle. Construct an angle of  $60^\circ$  at vertex A and another one at vertex B. Produce the arms of the angles at A and B till they meet and we have an equilateral triangle ABC. See Figure 1(b).

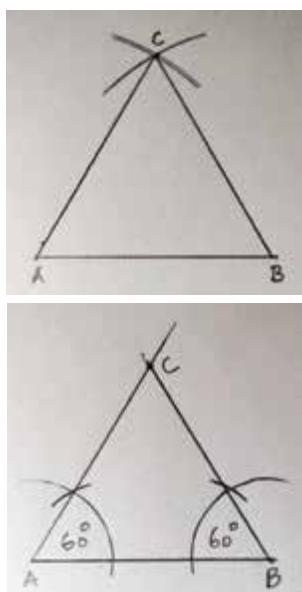


Figure 1. Compass and ruler methods of constructing an equilateral triangle: (a) using the property of equality of sides, (b) using the property of equality of angles

Although both methods produce equilateral triangles, students are often unable to provide a reasoning or justification as to why these methods work. Method 1 uses the property that all sides of an equilateral triangle are equal while method 2 is based on the fact that all angles have a measure of  $60^\circ$  each. However, these properties may not be visually evident while following the “steps of construction” using a compass and ruler.

A group of grade 6 students were given the opportunity to explore the construction of geometrical figures using GeoGebra. As they did not have any prior exposure to using GeoGebra, they underwent a 90-minute session in which they were familiarized with the basic construction tools and were also given time to explore these tools. Following this, they were required to construct two figures using GeoGebra – an equilateral triangle and a regular hexagon.

While constructing an equilateral triangle some interesting initial responses were elicited by students. Some used the **Polygon** tool to draw a triangle (Figure 2(a)) and then tried to make it an equilateral triangle through manual adjustment, that is, by dragging one of the vertices and adjusting the side lengths. It was pointed out by their teacher that if by dragging a vertex, the triangle loses its properties, that is equality of sides and all angles being  $60^\circ$ , then the construction is not a robust one. At this point students were introduced to the idea of a “drag test” in GeoGebra. The drag test is a dragging strategy that can facilitate generalization since its purpose is to let a DGS user discern or verify invariants under varying appearances of the object of exploration. A robust or correctly constructed figure, which does not get “messed up” upon dragging is said to survive the “drag test”. Another group of students used the **Regular polygon** tool to construct an equilateral triangle. This indeed led to a robust equilateral triangle, which did not lose its properties upon dragging (Figure 2(b)). The Algebra view was used to verify that all sides were equal and all angles measure  $60^\circ$ . However, a few other students felt that this was “cheating” as the **Regular polygon** tool was an inbuilt feature of GeoGebra and therefore the figure that emerged was not actually a constructed one! There was some difference of opinion among students on this matter. Taking advantage of the situation, the teacher then asked the students to construct an equilateral triangle using GeoGebra’s construction tools. She prompted students to use the **Segment** tool, **Circle** tool, and **Angle** tool for completing the construction.

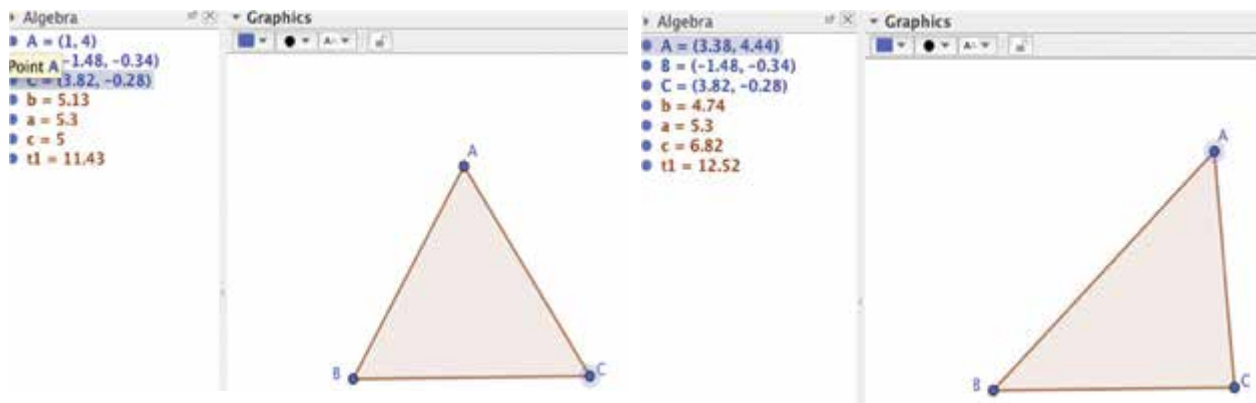


Figure 2(a) Students used the **Polygon** tool to draw an equilateral triangle via manual adjustment. However, on dragging a vertex the triangle was no longer equilateral.

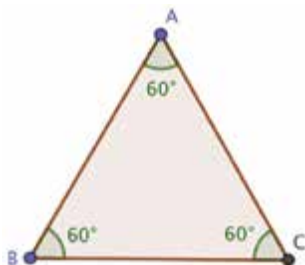


Figure 2(b) The Regular polygon tool led to an equilateral triangle, which retained its properties upon dragging.

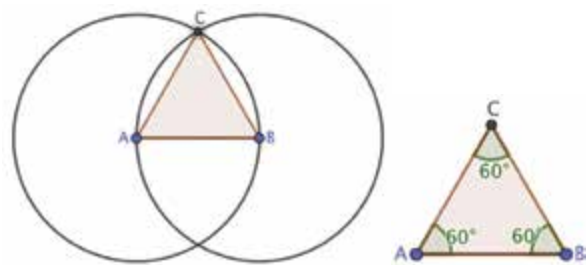


Figure 3(a) Students used the **Segment** tool, **Circle** tool and **Polygon** tool to construct an equilateral triangle. The hide option was used to remove the circles and the **Angle** tool was used to verify that all angles measure  $60^\circ$ .

After this the students at once set to work in pairs. Most of their constructions may be classified under one of the following two approaches.

*Approach 1:* Students drew a line segment AB using the *Segment* tool. Using A as centre and AB as radius, a circle was drawn using the *Circle with centre and point* tool (available within the *Circle* tool). Similarly, using B as centre and AB as radius, another circle was drawn which intersected the previous circle. One of the intersection points of the two circles was selected as the third vertex of the triangle. The *Polygon* tool was then used to complete the triangle (See Figure 3(a)). When asked for the rationale for this approach, some students expressed that they were trying to imitate what they would do using a compass and ruler on paper. However, instead of drawing arcs, they were drawing circles. A student argued, “drawing full circles helps to

see that all three sides are equal since these sides are radii of equal circles”. It was heartening to see that students were eagerly providing explanations for the way they chose to proceed with their construction.

*Approach 2:* Students drew a line segment AB using the **Segment** tool. At A, a  $60^\circ$  angle was constructed using the **Angle with given size** option (available within the **Angle** tool). Similarly, a  $60^\circ$  angle was also constructed at B. However, it took students some time to understand the clockwise and anticlockwise orientations while constructing the angles. Once this was done, they were able to identify the third vertex of the triangle. See Figure 3(b). They also measured the third angle to ensure that it had a measure of  $60^\circ$ .

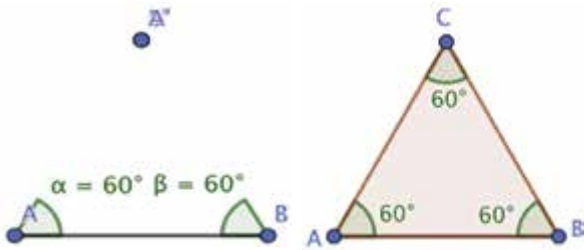


Figure 3(b) The **Angle with given size** option was used to construct angles of  $60^\circ$  at vertices A and B respectively and the **Polygon** tool was used to complete the equilateral triangle.

Both approaches led to equilateral triangles, which survived the drag test. Thus by dragging a vertex of the constructed triangle, they could modify the side lengths and the orientation of the triangle, but its fundamental properties, namely equal sides and equal angles remained invariant in a dragging episode. This led them to experience varying appearances of an equilateral triangle in a visual-dynamic way. While dragging parts of a figure, the **Algebra view** as well as the **Graphics view** revealed those attributes, which vary and those, which remain invariant. Being able to discern what varies and what remains invariant is key to experiencing a mathematical property. According to Leung (2003)

*A key feature of DGS is its ability to visually represent geometrical invariants amidst simultaneous variations induced by dragging activities...*

*...when engaging in mathematical activities or reasoning, one often tries to comprehend abstract concepts by some kind of “mental animation”, that is, mentally visualising variations of conceptual objects in the hope of “seeing” patterns of variation or invariant properties (p. 197)*

The next task posed to the students was to construct a regular hexagon (without using the **Regular polygon** tool). Before beginning the task, students were encouraged to list out the properties of a regular hexagon, which would help them in their construction. Thus, all sides are equal, all interior angles measure  $120^\circ$ , exterior angles measure  $60^\circ$  and that

a regular hexagon can be divided into six equilateral triangles were pointed out as some of the properties of a regular hexagon. The teacher asked them to use these properties in conceptualizing their construction process. The end result was to be a hexagon, which would survive the drag test.

The hexagon construction led to many interesting responses. One group of students used the fact that every interior angle measures  $120^\circ$ . They drew a line segment AB and used the **Angle with given size** from the **Angle** tool menu to draw angles of  $120^\circ$  both at A and B. They were familiar with the clockwise and counterclockwise orientation from the equilateral triangle construction earlier. As new vertices of the hexagon emerged, they constructed more interior angles of  $120^\circ$  and finally completed the hexagon. See Figure 4(a).

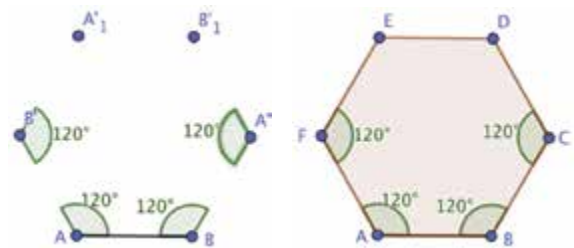


Figure 4(a) The **Angle with given size** option was used to construct a regular hexagon.

Another group wanted to use the fact that a regular hexagon comprises six equilateral triangles. These students began the construction with an equilateral triangle following which they used the **Reflect about Line** option to reflect the triangle about its sides. Repeating the reflection process, they completed the hexagon. See Figure 4(b).

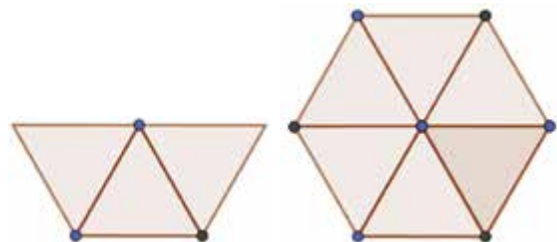


Figure 4(b) The **Reflect about Line** option was used to construct a hexagon. An equilateral triangle was constructed and was reflected about one of its sides.

A third group of students constructed six intersecting circles of the same radii. After this, they selected the six points of intersection of the pairwise intersecting circles to construct a hexagon. Later they used the hide/unhide feature to remove the circles leaving on the screen a regular hexagon. See Figure 4(c).

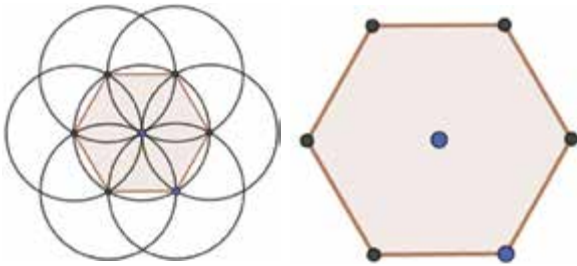


Figure 4(c) Students constructed a hexagon by joining the points of intersection of six intersecting circles of the same radii.

It was interesting to see that students could approach the construction of the hexagon using their own reasoning and their knowledge of GeoGebra tools. Feedback taken at the end of the session revealed that most students considered the use of GeoGebra “very interesting” and “more accurate than compass and ruler constructions”. They also opined that GeoGebra enabled them to construct a figure in multiple ways. Further, they expressed that constructing a regular hexagon was a novel experience for them. A student also commented that the GeoGebra method of constructing the hexagon would also enable her to do the construction using compass and ruler.

### Constructing a square

In another study, grade 7 students were asked to construct a square using GeoGebra. Although they had constructed squares and rectangles in their regular classes using compass and ruler, they had no prior experience in using GeoGebra. After going through a 90-minute session on exploring the basic construction tools, their initial response was to draw a square using the **Polygon** tool through manual adjustment (See Figure 5(a)). However, upon dragging a vertex, the figure would lose its properties and it would no longer be a square. The students

then discovered the **Regular polygon** tool. Using this they were able to construct a robust square, which survived the drag test. (Figure 5(b)). Later, when the researchers asked the students to construct a square using GeoGebra’s construction tools, they appeared to use their prior knowledge of constructing a square using a compass and ruler to figure out the construction process in GeoGebra. One of their approaches was to construct perpendiculars at end points A and B of a line segment AB using the **Perpendicular line** tool. Students further recalled that arcs had to be drawn in the compass and ruler method. When asked why the arcs were required, one student said “the arc is needed to mark the point on the perpendicular line so that the sides of the square remain equal to the line segment AB.” After this, they used the **circle tool** to complete the construction of the square (See Figure 5(b)). Any errors in the construction process were rectified using the hide/unhide option. In fact, the GeoGebra construction required more mathematical knowledge about the properties of a square and circle than the compass-ruler construction. Contrasting between the static construction using compass and ruler and the dynamic construction on GeoGebra, a student commented “when we construct a square using compass and ruler, like the way our teacher taught us, we cannot do anything to it afterwards. It cannot be changed unless we use an eraser. But this GeoGebra square can be moved and changed in both position and in size and even after dragging it remains a square!” When asked to check if the constructed figure is actually a square, students resorted to the **Algebra view** and verified that all the sides were equal and all angles were right angles. By dragging, students learnt that the properties of the square which are relevant to the construction process are equality of sides and all angles being right angles (which should remain invariant) whereas the side length of the square or its orientation are irrelevant (and may vary). This illustrates well the “drag for contrast approach” by which students were able to construct a ‘robust’ square, which ‘survived the drag test’. According to Leung (2002):

Contrast is about seeing differences, comparing between what is and what is not, hence anticipating (conjecturing) what can be and cannot be (p. 6).

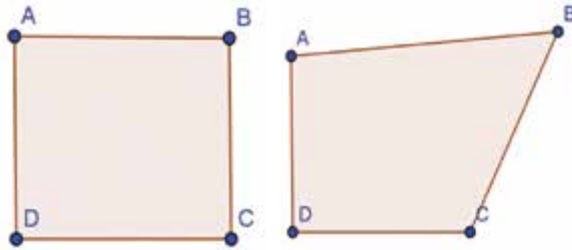


Figure 5(a) A square drawn via manual adjustment does not retain its properties if one of its vertices is dragged.

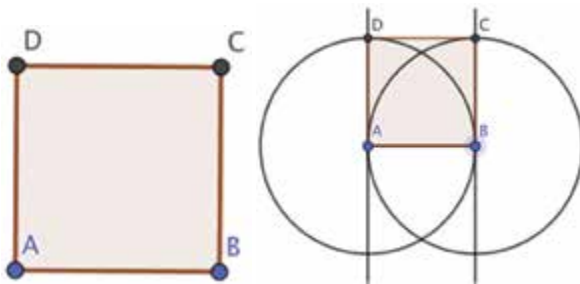


Figure 5(b) Squares constructed using the **Regular Polygon** tool and tools such as **Perpendicular line**, **Circle with Centre and Point** lead to robust squares, which survive the drag test.

### Constructing a rectangle

As a natural extension to the square construction activity, the grade 7 students wanted to construct a rectangle. The researcher asked them to construct a rectangle using GeoGebra's construction tools. Students felt that this was an easy task as it would be very similar to constructing a square. They drew a line segment AB using the **Segment** tool and then used the **Perpendicular line** tool to draw perpendiculars to AB at A and B respectively. Once this was done, they realized that the next vertex, C, of the rectangle can be chosen anywhere on one of the perpendiculars. Once C is chosen, a line parallel to AB was drawn through C. The **Intersect** tool was then used to identify the fourth vertex D (see Figure 6(a)). Hiding the perpendicular and parallel lines led to a robust rectangle, which maintained its properties upon dragging. Students observed that in this

“draggable dynamic rectangle” (a term coined by the students), the opposite sides are equal and all angles are right angles. The researcher asked if the rectangle could be dragged into a square. Students dragged one of the vertices A, B or C and observed that the rectangle can indeed be made into a square. See Figure 6(b). However one student remarked that the square they had constructed earlier cannot be dragged into a rectangle. This led to a discussion and students concluded that a rectangle could be made into a square by dragging, since a square is a special kind of a rectangle. However, once a square is constructed it cannot be dragged into a rectangle. It was interesting to see that students' explanations were moving towards the relationship between figures and hierarchical inclusions.

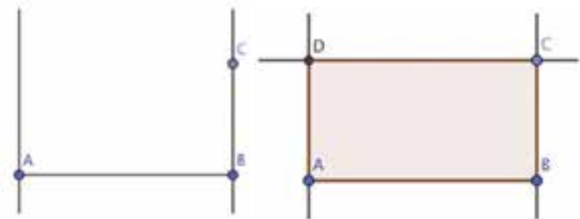


Figure 6(a) A rectangle was constructed using the **Perpendicular Line** and **Parallel Line** tools.

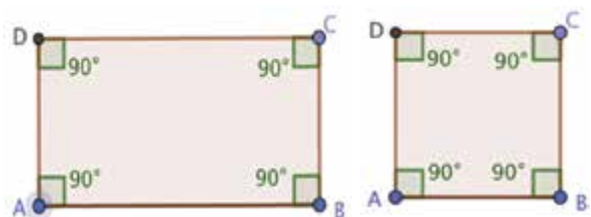


Figure 6(b) Students verified the properties of the rectangle and also dragged its vertices to form a square.

### Constructing a rhombus

Asking students to produce constructions, rather than drawings, can help improve their understanding of formal definitions and relationships among geometric objects. A group of grade 8 students who had studied the topic on quadrilaterals were assigned the following task.

Task: Construct a rhombus using GeoGebra and explain why it is a rhombus. Reflect on how this is different from constructing a square.

This group of students already had the prior experience of constructing a square using GeoGebra. They were also familiar with the properties of a rhombus, namely, equality of sides, equality of opposite angles and that diagonals bisect each other at right angles. After recalling these properties students decided that it would be easier to begin the construction of the rhombus with one of its diagonals. A line segment AB was drawn using the **segment tool**. This was followed by constructing its **Perpendicular bisector** (since the other diagonal would be perpendicular to AB). The next step was to select the two vertices (of the rhombus) say C and D, on this perpendicular line, so that the sides AC, BC, AD and BD are equal. Many students selected a point C on the perpendicular line and joined it to the vertices A and B (See Figure 7(a)). A student remarked “As C is on the perpendicular bisector of AB, it will be equidistant from the points A and B”. Following this, they tried to identify a point D on the perpendicular line so that C and D were on opposite sides of AB. Joining D to A and B led to a rhombus like figure. However this would not always remain a rhombus if the point D was dragged on the perpendicular line. (See Figure 7(a)). Students began to explore a way of working around this problem. They realised that the points C and D should be equidistant from the diagonal AB. After some exploration, a student came up with the suggestion “let’s select a point C on the perpendicular line and then use the **Reflect about Line** tool to reflect C about the line AB”. Indeed this led to a point C', the image of C on line AB. This suggestion was appreciated by everyone and it was decided that C' could be renamed as D. After completing the construction, students used the **Angle** tool to measure the angles of the rhombus and also used the **Algebra view** to check if all the sides were equal. This led to a dynamic rhombus, which preserved its properties upon dragging. See figure 7(b).

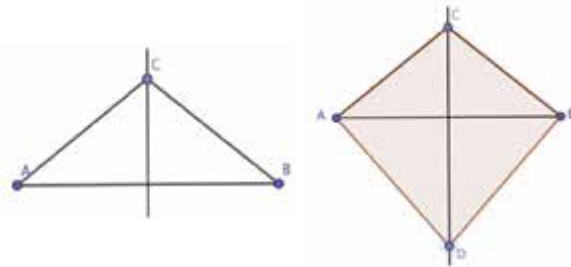


Figure 7 (a) Students’ approach of drawing a rhombus in GeoGebra. By dragging the point D the figure was no longer a rhombus.

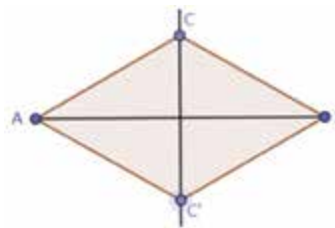


Figure 7(b) Students used the **Reflect about Line** tool to reflect the point C about AB to construct a rhombus.

At this point the teacher decided to introduce students to the use of sliders to construct a rhombus. Two sliders were created. One (named *side*) was used to vary the side length of the rhombus and the other (named  $\alpha$ ) was used to vary the acute angle (marked in red in Figure 7(c)). By dragging the blob on the sliders, the angles and side lengths of the rhombus could be varied. However, the equality of opposite sides and opposite angles remained invariant. Three examples of rhombuses which emerged from dragging the sliders are shown in Figure 7(c).

This aptly illustrates the *mathematical variability principle* by Zoltan P. Dienes (1963), which states that

as every mathematical concept involves variables, all these mathematical variables need to be varied if the full generality of the mathematical concept is to be achieved. (p. 158)

In Figure 7(c) dragging the sliders can be used to vary the irrelevant properties of the rhombus, namely side length and angle measures.

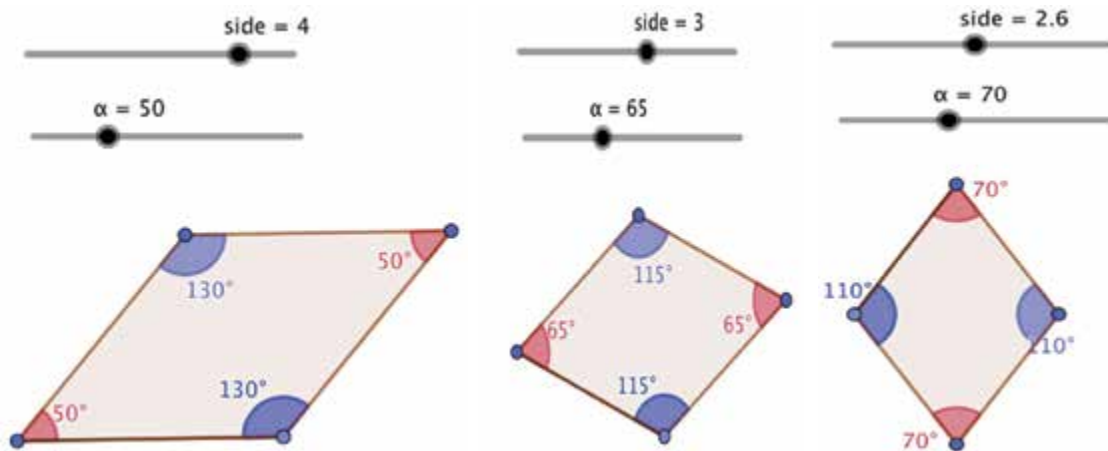


Figure 7(c) Sliders were used to construct a dynamic rhombus in which the length of sides and angle measures could be varied, while the equality of sides and equality of opposite angles remained invariant.

However, in the dragging episode, the equality of sides and equality of opposite angles remain invariant. This contrasting experience of the varying and the invariant enables the user to experience the mathematical concept of the rhombus in its fully general form.

### Conclusion

Typically, in the traditional classroom, students are asked to construct geometrical figures using a sequence of steps. This article attempts to provide the reader with alternative ways of constructing the familiar geometric figures. The dynamic nature of the tool allows students to

explore the properties of figures and also use them in the process of construction. From a pedagogical perspective, dynamic geometry software has created new interest in teaching of geometry as it enables students to perform geometric constructions with a high degree of accuracy and also explore constructions in multiple ways using the dragging tool. Experiencing geometrical figures in a visual-dynamic way facilitates the understanding of propositions, making and testing of conjectures, provide explanations and to engage in argumentation finally leading to proof. The reader may wish to explore the construction of other geometrical figures using GeoGebra.

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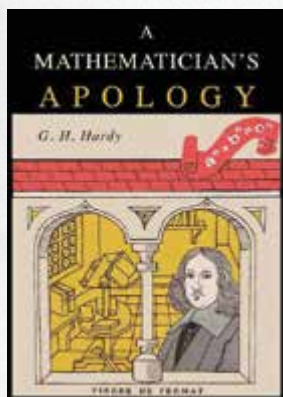
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JONAKI GHOSH is an Associate Professor in the Dept. of Elementary Education, Lady Sri Ram College, University of Delhi where she teaches courses related to math education. She obtained her PhD in Applied Mathematics from Jamia Milia Islamia University, New Delhi, and her M.Sc. from IIT Kanpur. She has taught mathematics at the Delhi Public School, R K Puram, where she set up the Math Laboratory & Technology Centre. She has started a Foundation through which she conducts professional development programmes for math teachers. Her primary area of research interest is in the use of technology in mathematics instruction. She is a member of the Indo Swedish Working Group on Mathematics Education. She regularly participates in national and international conferences. She has published articles in journals and authored books for school students. She may be contacted at [jonakibghosh@gmail.com](mailto:jonakibghosh@gmail.com)

# Book Review: A Mathematician's Apology, by G.H. Hardy

*Reviewed by Paraj Modi*



The title 'A Mathematician's Apology' strikes the readers unusually. Why is a mathematician as great as G.H. Hardy himself, apologising? The entire book is but an apology, which aims to offer a defence in pursuit of mathematics. Published in 1940, this apology stands relevant in most respects even today. Originating from one of the finest mathematicians as Hardy, this apology invokes realisation of passion in one's conscience, especially in students as myself.

Hardy begins by proclaiming that writing this book is nothing but a '*melancholy experience*'. His justification of this phrase explains the true pursuit of a mathematician - '*to add to mathematics, and not to talk about what he or other mathematicians have done.*' The entire book is written with a striking element of truth and utmost honesty. Hardy unhesitatingly calls this piece of writing a '*confession of weakness*' of a mathematician which may provoke scorn by other mathematicians.

Hardy adopts a unique approach throughout the book - wherein the readers can feel as if Hardy is talking to them. He asks questions, analyses what their answers may be, opines on them himself and then leaves it to the readers to understand the ideas as they like. Hardy questions, "*Is mathematics 'unprofitable'?*" He then goes on to explore the possible answer to it, "*In some ways, plainly, it is not; for example, it gives great pleasure to quite a large number of people. I was thinking of 'profit', however, in a narrower sense.*" He ends by specifying his notion of profit, thereby prompting readers to think.

Hardy believes that mathematics is a '*young man's game*' — that age is indeed an influential factor in determining the success of any mathematician. He aids his claim through the life stories of Galois, Abel, Ramanujan, Riemann, etc. Although age might be a limiting factor, Hardy does not forget to mention the permanence and

*Keywords: Student readers, mathematicians, introspection*

timelessness of mathematical achievement. He writes, *“In these days of conflict between ancient and modern studies, there must surely be something to be said for a study which did not begin with Pythagoras, and will not end with Einstein, but is the oldest and youngest of all.”* Somewhere, the transcendental nature of mathematics is being reflected here. We are divided by boundaries, but an art form, as beautiful as mathematics, connects us all universally and appeals to all equally.

This piece of work is essentially relevant for fellow students in high school, because Hardy’s reflections on what the youth should do, are particularly substantial. He says that the youth must be ambitious — it is after all, this ambition that drives all worthy discoveries or inventions - whatever their area of interest might be. Moreover, this book brings along with it an excellent opportunity to interpret what beauty of any sort may be like. Hardy fondly mentions, *“It may be very hard to define mathematical beauty, but that is just as true of beauty of any kind — we may not quite know what we mean by a beautiful poem, but that does not prevent us from recognising one when we read it.”* Through such assertions, Hardy forces the readers to appreciate the aesthetic appeal of mathematics.

Hardy goes on to explain what, according to him is the ‘utility’ of mathematics, and what ‘significant’ mathematics is. This analysis was my most favourite part in the entire book. Hardy says that any serious mathematical idea must have *depth* and *generality*. These two characteristics may sound exactly opposite, but indeed, Hardy has an argument to make!

Drafted across several separate essays, the book is not only an insightful read, but also a spiritual experience any art lover would love to have. As Hardy constantly draws analogues between chess, mathematics and poetry, the book has something

to offer to everybody. While most ordinary readers may think that this book would have a lot to deal with complex mathematics — it is an utter misconception. This book is but a brief introduction meant for common people to dive into the realms of divinity through mathematics.

Hardy was undoubtedly a fabulous mathematician, but this book also proves him to be a lucid writer - who expresses his thoughts with unflinching clarity and paramount honesty. He has a rhythmic knack of writing when he writes, *“Chess problems are the hymn-tunes of mathematics”* or *“A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.”* These philosophical notes portray Hardy as a devoted thinker with much depth.

But the mystery persists — what is the apology exactly for? During one of the insightful discussions with my mathematics teacher, he expressed that the apology is to the people who are not really fascinated by mathematics. Hardy, as a torch-bearer of the mathematics community, apologises to those people as they cannot perceive the profound inexplicable beauty of mathematics. As Hardy likes to say it, *“Immortality may be a silly word, but a mathematician has the best chance of whatever it may mean.”*

### Post Script from the Reviewer

I am grateful to my mathematics teacher for recommending this and other books like *‘Fermat’s Last Theorem’* by *Simon Singh*, and *‘Uncle Petros and the Goldbach’s Conjecture’* by *Apostolos Doxiadis*. Just as I have thoroughly enjoyed reading these literary marvels pertaining to mathematics, I believe the youth of my age would enjoy doing so as well. This would not only broaden perspectives but also bring clarity regarding the choice of one’s career.



**PARAJ MODI** is a student who has just completed 12th grade from Maharaja Agrasen Vidyalaya, Ahmedabad. She is passionate about biology, physics and mathematics. She plays the Tabla and loves to explore mathematical patterns in Indian Classical Music, Nature and Origami. She is a poet and has published a poetry book ‘Paradise Out of Words’. Above all, she is a curious person who loves to learn and discuss. She may be contacted at [parajmodi2703@gmail.com](mailto:parajmodi2703@gmail.com)

# Manipulative Review: Rangometry

*Reviewed by Aaloka Kanhere*

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In this article I would like to review an extremely colourful and useful learning aid called Rangometry. This teaching-learning aid was introduced by Jodo Gyan, Delhi. This is a very powerful tool that can be used for various classroom activities such as story-telling, counting, visualizing patterns and understanding shapes and angles.



The kit consists of various colourful pieces. A closer look reveals that there are eight different shapes and several copies of these shapes in different colours. Out of these eight shapes, 3 shapes are regular polygons; an equilateral triangle, a square and a hexagon. The rest of the shapes are irregular polygons; an isosceles triangle, two rhombi, a rectangle, and a trapezium.

The lengths of each of these are chosen very carefully and each of these shapes have at least one side of the same length. All

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*Keywords: Learning through play, shapes, angles, sides, counters, patterns, pictures*


the three regular polygons have sides of equal length. Moreover, all the sides of the square and the rhombi are of equal length and differ only in their angles. Similarly, the shorter side of the rectangle is of the same length as the side of the square.



My first experience with Rangometry was with very young children. When the kit was introduced to the children, they immediately took to it due to the bright colours. Children made lots of different designs from the pieces and then started developing stories around them in groups. One of the older children agreed to write up the story for each group. Thus began a very colourful session of story-telling and writing stories.

Several interesting activities involving counting can be done with these pieces. It can begin with children counting the number of pieces they used to make a design or an object. Similarly, the children can be asked to pick a certain number of pieces, or they can draw numbered chits and make shapes with a specific number of objects. Such counting activities are very meaningful to the children and feel purposeful.

Name	Number of pieces picked up	Design made
Shivani	6	Flower
Anjali	10	House
Jitendra	7	Plane

Another interesting activity that can be done with this kit is of visualizing patterns. So, patterns like these  can be made using this kit and based on the level of the students, one can ask questions like, ‘What will the 9th shape be?’ or ‘What will the colour of the 50th shape be?’



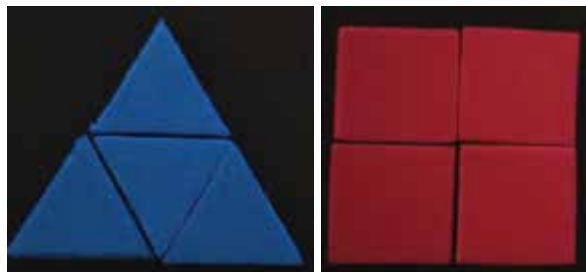
For students from higher classes, patterns involving ‘known’ numbers can also be made. So instead of the number pattern 1, 4, 9, ..., one can use the squares to make bigger squares like these which also makes one understand why they are called square numbers.



### Making shapes from shapes

Making similar shapes from the pieces is another interesting activity that can be done with the kit. Can you make a triangle using triangles? Can you make a square from squares? What about hexagons from the hexagons?

### Making triangle from triangles and square from squares



During this activity, interesting discussions can be held in the classroom. Like why is it possible to make a square from squares but not possible to make a hexagon from hexagons?

A very interesting exercise that this kit offers is making different polygons from the given shapes. In one such exercise with teachers, the teachers were asked to make hexagons using these pieces. All of them came up with the following configurations. And they concluded that only one type of hexagon can be made with these pieces.



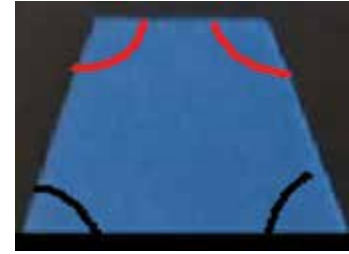
When asked if there were any other hexagons possible apart from the regular ones, all of us realized that when we think about hexagons we picture only regular hexagons, a common mistake made by children too.

Once one explores the idea of making different kinds of irregular hexagons, one finds that a lot of hexagons can be made using these pieces. Some of the hexagons are shown below.



Like hexagons, children can also be asked to make pentagons or octagons. Such exercises help children think of polygons beyond the regular polygons. There can also be a discussion on whether one can find a pattern in making these polygons. For example, in hexagons, one quadrilateral and two triangles can make a hexagon, or two quadrilaterals can make a hexagon.

This kit can be a very useful tool to understand angles. To start with, one can try to arrange the angles in each piece in ascending or descending order of the measures of their interior angles. It is also interesting to discover which angles in each polygon are the same and which are different. Like in the case of a trapezium. There are two angles which are acute and two angles which are obtuse. The acute angles are equal and the obtuse angles are also equal to each other. See the image given below. Angles marked in black are acute and equal and ones marked in red are obtuse and equal.




Children would have to devise strategies to do this comparison in the absence of a protractor.

The first five angles are arranged in ascending order in the next picture.

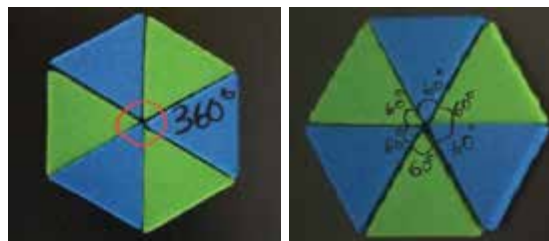


The activity I find the most interesting is the activity to find the measures of angles of the various pieces in the kit without using a protractor. This activity can be very enriching and can help children develop a strong understanding of angles.

Using just the fact that the sum of angles around a point is 360 degrees, one can find measures of all the remaining angles.

Like, one can establish that the measure of all angles of this piece  is 60 degrees.

After one knows the measures of some angles, one can find the remaining angles. One example is given in the next image.



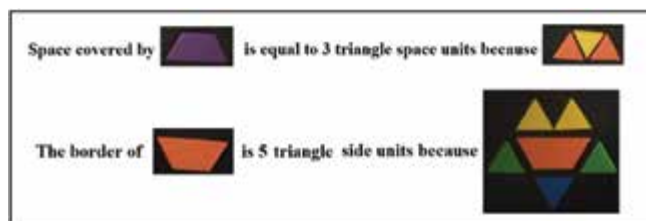
During this activity, there are ample opportunities to talk about angles and their properties like adding angles, subtracting one angle from another, or comparing angles.

One concept that can be explored using this kit is that of area and perimeter. Children can be asked to make a design and then they can be asked to compare each of their designs and decide which is the biggest. This question can create a situation in the classroom where first the children would try to decide among themselves what is the measure of bigness, the number of pieces used or the space covered, or how big the border is. This is also an opening for teachers to start talking about informal units of area or length.



What is bigger? The flower or the lamp-shade?

The design of the kit enables us to use the side of an equilateral triangle as an informal unit to measure boundary or perimeter. Similarly, space covered by an equilateral triangle can be called 1 triangle space unit or 1 triangle area unit, and then one can measure shapes or designs made and children can describe them.



Such hands-on exercises to explore area or perimeter can prove to be very useful when they come across area and perimeter concepts.

Rangometry can be used for a wide range of concepts in school mathematics. For more details about the kit: <https://jodogyam.org/activity-resources-primary-rangometry/>

**Editor's Note:** Rangometry was introduced by Jodo Gyan and is now sold by various groups including Navnirmiti. Since these are made of Ethyl Vinyl Acetate (or EVA), they stick to boards (black, white or green) when wet and thus make a great display for the entire class. However, some other groups did make similar kits out of wood. The kits may be easily replicated with any cardboard or card type material as well.

Children are very good at thinking outside the box, usually better than the adults. So, they can use the rangometry kit to create a 3D picture, design, layout etc. which goes much beyond the intended objectives of the adults who designed this!

Also, Mathigon polypad (<https://mathigon.org/polypad#polygons>) includes similar pieces and more. While the 3D aspect is lost, the user can enjoy the benefits of this virtual resource including but not limited to (i) unlimited pieces that can be arranged and (ii) choice of the colour of each piece.



**AALOKA** is a mathematics educator. Currently, she works as a freelancer but she has been associated with the Homi Bhabha Center for Science Education, Mumbai, and Eklavya Foundation, Madhya Pradesh, in the past. Aaloka may be contacted at [aalokakanhere@gmail.com](mailto:aalokakanhere@gmail.com)

# A Problem from the Putnam 2022 Competition

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MURALIDHAR RAO &  
PARINITHA M

In this article, we discuss a problem on polynomials adapted from the Putnam competition of 2022.

**Problem 1.** Let  $n$  be an integer with  $n \geq 2$ . Over all real polynomials  $p(x)$  of degree  $n$ , what is the largest possible number of negative coefficients in  $(p(x))^2$ ?

**Developing the strategy.** To develop an insight into the solution, we study the cases  $n = 2$  and  $n = 3$ . We then use our observations to conjecture a bound for the number of negative coefficients in  $(p(x))^2$ , and we then prove the conjecture by a general argument. We finally construct a polynomial  $p(x)$  of degree  $n$  with the largest possible number of negative coefficients in  $(p(x))^2$ .

**Solution.** First, observe that the coefficients of  $x^n$  and  $x^0$  in  $(p(x))^2$  are always positive. Hence, in order to maximize the number of negative coefficients in  $(p(x))^2$ , let us see if it is possible for all the remaining coefficients to be negative.

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*Keywords: Putnam competition, polynomials, coefficient, mathematical induction*

Consider the case for  $n = 2$ . Let

$$p(x) = a_2x^2 + a_1x + a_0,$$

where, without any loss of generality, we may assume that  $a_0 \geq 0$ , because  $(p(x))^2 = (-p(x))^2$ . Then

$$(p(x))^2 = (a_2x^2 + a_1x + a_0)^2 = a_2^2x^4 + (2a_1a_2)x^3 + (2a_0a_2 + a_1^2)x^2 + 2a_0a_1x + a_0^2.$$

The coefficients of  $x^4$  and  $x^0$  are non-negative. Let us see if the coefficients of  $x$ ,  $x^2$ ,  $x^3$  can all be negative.

If the coefficient of  $x$  is negative, i.e.,  $2a_0a_1 < 0$ , then as  $a_0 \geq 0$ , it follows that  $a_1 < 0$ .

If the coefficient of  $x^2$  is negative, i.e.,  $2a_0a_2 + a_1^2 < 0$ , then it follows that  $a_2 < 0$ . But then, the coefficient of  $x^3$  will be positive, since  $2a_1a_2 > 0$ .

We observe that not all of the coefficients of  $x$ ,  $x^2$ ,  $x^3$  can be negative.

Therefore, for  $n = 2$ , the number of negative coefficients in  $(p(x))^2$  cannot exceed 2.

Next, consider case  $n = 3$ . Let

$$p(x) = a_3x^3 + a_2x^2 + a_1x + a_0,$$

with  $a_0 \geq 0$ . Then

$$\begin{aligned} (p(x))^2 &= (a_3x^3 + a_2x^2 + a_1x + a_0)^2 \\ &= a_3^2x^6 + 2a_3a_2x^5 + (a_2^2 + 2a_3a_1)x^4 + (2a_2a_1 + 2a_3a_0)x^3 \\ &\quad + (2a_0a_2 + a_1^2)x^2 + 2a_0a_1x + a_0^2. \end{aligned}$$

If coefficient of  $x$  is negative, i.e.,  $2a_0a_1 < 0$ , then  $a_1 < 0$ .

If the coefficient of  $x^2$  is negative, i.e.,  $(2a_0a_2 + a_1^2) < 0$ , then  $2a_0a_2 < 0$ . As  $a_0 \geq 0$ , we must have  $a_2 < 0$ . Now, if the coefficient of  $x^3$  is negative, i.e.,  $2a_2a_1 + 2a_3a_0 < 0$ , then we must have  $a_3 < 0$ . Thus, we have  $a_2 < 0$ ,  $a_3 < 0$ , so  $2a_2a_3 > 0$ . We see that the coefficients of  $x^4$  and  $x^5$  are positive.

So not all the coefficients of  $x$ ,  $x^2$ ,  $x^3$ ,  $x^4$ ,  $x^5$  can be made negative.

Therefore, for  $n = 3$ , the number of negative coefficients in  $(p(x))^2$  cannot exceed 4.

Based on these observations, we make a guess that for a polynomial  $p(x)$  of degree  $n$ , the number of negative coefficients in  $(p(x))^2$  cannot exceed  $2n - 2$ .

We establish this claim in the next section.

**Bound for the number of negative coefficients in  $(p(x))^2$ .** Since the coefficients of  $x^{2n}$  and  $x^0$  in  $(p(x))^2$  are always positive, it suffices to show that we cannot have every other coefficient negative.

Suppose that this is the case; i.e., the coefficients of  $x^{2n-1}$ ,  $x^{2n-2}$ ,  $\dots$ ,  $x^2$ ,  $x^1$  are all negative.

Write  $p(x) = a_nx^n + \dots + a_1x + a_0$ ; without loss of generality, let  $a_0 > 0$ . We start by proving, using induction, that  $a_1, a_2, \dots, a_n$  must all be negative.

The base case of  $a_1 < 0$  is true because otherwise the coefficient of  $x$  in  $(p(x))^2$  would be non-negative.

Now suppose that  $a_1 < 0, a_2 < 0, a_3 < 0, \dots, a_k < 0$  for some value of  $k$  with  $1 \leq k \leq n - 1$ . We shall prove that  $a_{k+1} < 0$  as well.

Let the coefficient of  $x^{k+1}$  in  $(p(x))^2$  be  $b_{k+1}$ ; then

$$b_{k+1} = 2a_0a_{k+1} + a_1a_k + a_2a_{k-1} + \cdots + a_ka_1.$$

Therefore,

$$2a_0a_{k+1} = b_{k+1} - (a_1a_k + a_2a_{k-1} + \cdots + a_ka_1).$$

By the inductive hypothesis, the summation in the bracket is positive, and by assumption,

$$b_{k+1} < 0.$$

As  $a_0 > 0$ , it follows that  $a_{k+1} < 0$ , completing the induction.

It follows that  $a_1, a_2, \dots, a_n$  are all negative.

But if  $a_1, a_2, \dots, a_n < 0$ , then the coefficient of  $x^{2n-1}$  in  $(p(x))^2$  (which equals  $2a_{n-1}a_0$ ) must be positive. Thus, we have a contradiction to the assumption that the coefficients of all the terms ( $x^{2n-1}$  through  $x^1$ ) of  $(p(x))^2$  are all negative.

This shows that the number of negative coefficients in  $(p(x))^2$  is  $\leq 2n - 2$ .

**Construction of an optimal polynomial.** We now show that there is a polynomial  $p(x)$  of degree  $n$ , with real coefficients, such that the number of negative coefficients in  $(p(x))^2$  is  $2n - 2$ .

That is, we show that there is a polynomial for which the bound proved above is attained.

To this end we consider:

$$p(x) = x^n - ax^{n-1} - ax^{n-2} - \cdots - ax^2 - ax + 1, \quad \text{where } a > 0.$$

We have not yet specified the value of  $a$  but we shall do so shortly. The above expression may be written as

$$p(x) = (x^n + 1) - a(x^{n-1} + x^{n-2} + \cdots + x).$$

Then,

$$\begin{aligned} (p(x))^2 &= (x^{2n} + 2x^n + 1) + a^2(x^{n-1} + x^{n-2} + \cdots + x)^2 \\ &\quad - 2a(x^n + 1)(x^{n-1} + x^{n-2} + \cdots + x) \\ &= (x^{2n} + 2x^n + 1) + a^2(x^{2n-2} + 2x^{2n-3} + 3x^{2n-4} + \cdots + (n-1)x^n \\ &\quad + (n-2)x^{n-1} + (n-3)x^{n-2} + \cdots + x^2) \\ &\quad - 2a(x^{2n-1} + x^{2n-2} + \cdots + x^{n+1} + x^{n-1} + x^{n-2} + \cdots + x) \\ &= x^{2n} + (-2a)x^{2n-1} + (a^2 - 2a)x^{2n-2} + \cdots + ((n-2)a^2 - 2a)x^{n+1} \\ &\quad + (2 + (n-1)a^2)x^n + ((n-2)a^2 - 2a)x^{n-1} + \cdots + (a^2 - 2a)x^2 + (-2a)x + 1. \end{aligned}$$

Now, if we choose  $a > 0$  such that  $2a > (n-2)a^2$  or, equivalently, such that

$$0 < a < \frac{2}{n-2},$$

then the coefficients of  $x^{2n}, x^n, x^0$  are positive and all the remaining coefficients are negative.

Thus, we have a polynomial  $p(x)$  of degree  $n$  such that the number of negative coefficients in  $(p(x))^2$  is  $2n - 2$ .

We conclude that among all real polynomials  $p(x)$  of degree  $n$ , the largest possible number of negative coefficients in  $(p(x))^2$  is  $2n - 2$ .  $\square$

We leave the following problem to the reader, as a challenge.

**Problem 2.** Let  $T$  denote the set of all polynomials with real coefficients of degree  $n$  such that all roots are real. As  $p(x)$  varies over  $T$ , what is the maximum number of negative coefficients in  $(p(x))^2$ ?

## References

1. William Lowell Putnam Mathematical Competition Problems, <https://www.maa.org/math-competitions/putnam-competition>



**MURALIDHAR RAO** has just passed Class 12 from The Learning Center PU College, Mangalore, Karnataka. He has a deep love for Mathematics, Physics, and Biology. He cleared the NTSE in Class 10, and did extremely well in the KVPY Examination 2021. He has qualified in the Indian National Math Olympiad and Indian National Astronomy Olympiad in the years 2021 and 2022. He has been selected for the Professor Harry Messel International Science School (University of Sydney, July 2023). He enjoys playing chess and listening to Indian Classical music. He may be contacted at [zugzwang186@gmail.com](mailto:zugzwang186@gmail.com).



**PARINITHA M** is a II year BSMS student at IISER, Tirupathi. She is deeply interested in Mathematics, especially Analysis, Number Theory, and Group Theory. She took part in the Simon Marais Math Competition 2022 and the Madhava Mathematics Competition 2023. She is a recipient of the DST Inspire scholarship. She is an active member of The Math Club at IISER and hopes to pursue educational activities related to mathematics. Her interests include classical instrumental music. She may be contacted at [parinitha.m.17@gmail.com](mailto:parinitha.m.17@gmail.com).

# Menelaus's Theorem

ANKUSH KUMAR  
PARCHA

**M**enelaus' theorem is an extremely important result in higher Euclidean geometry. We offer a proof of the theorem here using the sine rule from trigonometry.

## Menelaus's Theorem

In  $\triangle ABC$  a transversal line crosses the side lines  $CA$ ,  $AB$  and  $BC$  at points  $Q$ ,  $P$  and  $M$ , respectively. Then we have the following equality:

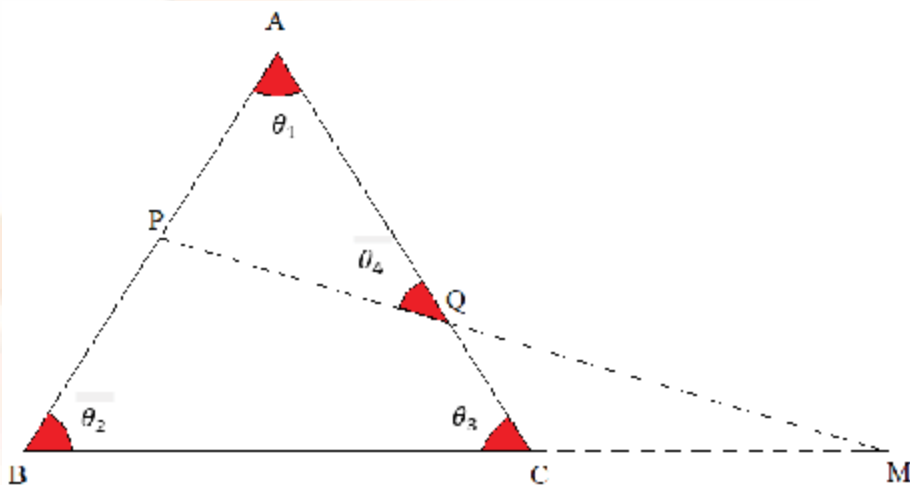
$$\frac{BM}{MC} \cdot \frac{CQ}{QA} \cdot \frac{AP}{PB} = 1.$$

## Proof

Apply the sine law in  $\triangle APQ$ :

$$\frac{\sin \theta_1}{PQ} = \frac{\sin \theta_4}{AP} = \frac{\sin (180^\circ - \theta_1 - \theta_4)}{QA},$$

therefore  $\frac{\sin \theta_4}{AP} = \frac{\sin (\theta_1 + \theta_4)}{QA} \implies \frac{AP}{QA} = \frac{\sin \theta_4}{\sin (\theta_1 + \theta_4)}.$



*Keywords: Euclidean geometry, Menelaus, proof, sine rule.*

Again, applying the sine law in  $\Delta QCM$ :

$$\frac{\sin \theta_3}{QM} = \frac{\sin \theta_4}{CM} = \frac{\sin (180^\circ + \theta_3 - \theta_4)}{CQ},$$

therefore  $\frac{\sin \theta_4}{CM} = \frac{\sin (\theta_3 - \theta_4)}{CQ} \implies \frac{CQ}{CM} = \frac{\sin (\theta_3 - \theta_4)}{\sin \theta_4}.$

Again, applying the sine law in  $\Delta PBM$ :

$$\frac{\sin (180^\circ + \theta_3 - \theta_4)}{PB} = \frac{\sin (\theta_1 + \theta_4)}{BM} = \frac{\sin \theta_2}{PM},$$

therefore  $\frac{\sin (\theta_3 - \theta_4)}{PB} = \frac{\sin (\theta_1 + \theta_4)}{BM} \implies \frac{BM}{PB} = \frac{\sin (\theta_1 + \theta_4)}{\sin (\theta_3 - \theta_4)}.$

Multiplying the corresponding sides of the three equalities, we get:

$$\frac{AP}{QA} \cdot \frac{CQ}{CM} \cdot \frac{BM}{PB} = \frac{\sin \theta_4}{\sin (\theta_1 + \theta_4)} \cdot \frac{\sin (\theta_3 - \theta_4)}{\sin \theta_4} \cdot \frac{\sin (\theta_1 + \theta_4)}{\sin (\theta_3 - \theta_4)},$$

therefore  $\frac{BM}{MC} \cdot \frac{CQ}{QA} \cdot \frac{AP}{PB} = 1.$

### Corollary

$$\left(1 + \frac{AP}{PB}\right) \left(1 + \frac{CM}{CB}\right) = \frac{QM}{QP} \cdot \frac{QA}{QC}.$$

### Proof

For  $\Delta MBP$  with  $AQC$  as transversal, Menelaus's Theorem can be written as

$$\frac{BA}{AP} \cdot \frac{PQ}{QM} \cdot \frac{MC}{CB} = 1.$$

Multiplying this with the previous result, we get:

$$\frac{BA}{AP} \cdot \frac{PQ}{QM} \cdot \frac{MC}{CB} \cdot \frac{BM}{MC} \cdot \frac{CQ}{QA} \cdot \frac{AP}{PB} = 1,$$

therefore  $\frac{BA}{PB} \cdot \frac{BM}{CB} = \frac{QM}{QP} \cdot \frac{QA}{QC} \implies \frac{BP + PA}{PB} \cdot \frac{BC + CM}{CB} = \frac{QM}{QP} \cdot \frac{QA}{QC}.$

Or:

$$\left(1 + \frac{AP}{PB}\right) \left(1 + \frac{CM}{CB}\right) = \frac{QM}{QP} \cdot \frac{QA}{QC}.$$



**ANKUSH KUMAR PARCHA** is currently studying BSc Physics Honours (second year) at Indira Gandhi National Open University (IGNOU), New Delhi. He has a deep interest in doing research in Physics and Mathematics. He has contributed many articles, problems, and solutions to magazines such as *Mathematical Gazette*, *AMJ*, *ISROSET*, *SSMJ*, *Pentagon*, *Octagon*, *At Right Angles* and *Mathematical Reflections*. He may be contacted at [ankushparcha11@gmail.com](mailto:ankushparcha11@gmail.com).

# Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

## A Call for Articles

Classroom teachers are at the forefront of helping students grasp core topics. Students with a strong foundation are better able to use key concepts to solve problems, apply more nuanced methods, and build a structure that help them learn more advanced topics.

The focal theme of this section of At Right Angles (AtRiA) is the teaching of various foundational topics in the school mathematics curriculum. In relation to these topics, it addresses issues such as knowledge demands for teaching, students' ideas as they come up in the classroom and how to build a connected understanding of the mathematical content.

Foundational topics include, but are not limited to, the following:

- Number systems, patterns and operations
- Fractions, ratios and decimals
- Proportional reasoning
- Integers
- Bridging Arithmetic-Algebra
- Geometry
- Measurement and Mensuration
- Data Handling
- Probability

We invite articles from teachers, teacher educators and others that are helpful in designing and implementing effective instruction. We strongly encourage submissions that draw directly on experiences of teaching. This is an opportunity to share your successful teaching episodes with AtRiA readers, and to reflect on what might have made them successful. We are also looking for articles that strengthen and support the teachers' own understanding of these topics and strengthen their pedagogical content knowledge.

Articles in this section may address key questions such as -

- What challenges did your students face while learning these fundamental mathematical topics?
- What approaches that you used were successful?
- What preparations, in terms of knowing mathematics, enacting the tasks and analysing students work were needed for effective instruction?
- What contexts, representations, models did you use that facilitated meaning making by your students?

**Send in your articles to**  
[AtRiA.editor@apu.edu.in](mailto:AtRiA.editor@apu.edu.in)

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## Policy for Accepting Articles

'At Right Angles' is an in-depth, serious magazine on mathematics and mathematics education. Hence articles must attempt to move beyond common myths, perceptions and fallacies about mathematics.

The magazine has zero tolerance for plagiarism. By submitting an article for publishing, the author is assumed to declare it to be original and not under any legal restriction for publication (e.g. previous copyright ownership). Wherever appropriate, relevant references and sources will be clearly indicated in the article.

'At Right Angles' brings out translations of the magazine in other Indian languages and uses the articles published on The Teachers' Portal of Azim Premji University to further disseminate information. Hence, Azim Premji University

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If the submitted article has already been published, the author is requested to seek permission from the previous publisher for re-publication in the magazine and mention the same in the form of an 'Author's Note' at the end of the article. It is also expected that the author forwards a copy of the permission letter, for our records. Similarly, if the author is sending his/her article to be re-published, (s) he is expected to ensure that due credit is then given to 'At Right Angles'.

While 'At Right Angles' welcomes a wide variety of articles, articles found relevant but not suitable for publication in the magazine may - with the author's permission - be used in other avenues of publication within the University network.

# The Closing Bracket . . .

This time we feature an innovative Teaching Learning Material SOGOL (सौ गोले) (Students Own Gadget for Outstanding Learning) designed and developed by Sunil Bajaj, Head of the Mathematics department SCERT, Haryana. (See Figure 1.)

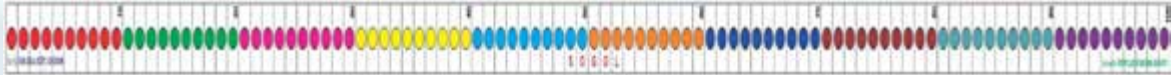


Figure 1

SOGOL is an approximately one metre long strip made of flex sheet, with dots arranged in colour blocks of ten. Each colour block occupies about 1cm which is further divided into 10 mm.

## SOGOL for Measurement

Since SOGOL is flexible, it may be used for measurement of straight and curved surfaces. We recommend that the 'Estimate and Measure' technique is used. Students can find out the area, perimeter, and circumference of real objects by measuring using the SOGOL strip.



Figure 2

## SOGOL for making shapes with fixed perimeter.

Children may be asked to make different shapes (squares/ rectangles/ triangles/circles etc.) with fixed perimeter or area using a Sogol strip.

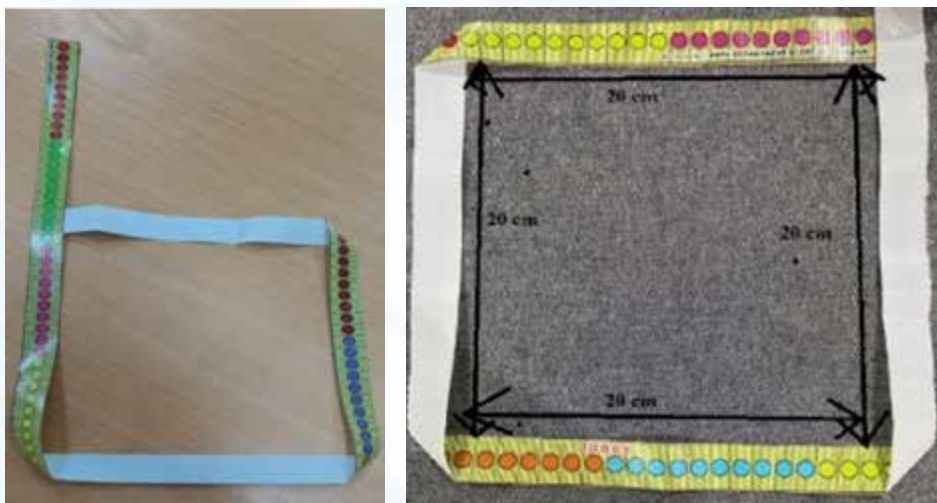


Figure 3: Square with Perimeter 80 cm (internal perimeter).

### SOGOL for Number Sense

This game can be played in pairs. One player from each pair starts rolling the SOGOL strip from a corner in a given time (for example, until the partner recites the numbers from 1 to 10). One who folds more will be the winner by counting dots or colour blocks, according to the age group. Initially they may count 1, 2, 3, 4 ..., but later in groups of 10.

### SOGOL for Number Operations

Divide the whole class into two groups. The group which reaches 20 (or any other pre-decided target such as 50 or 100) wins. The groups take turns to call out a number less than 4. As each number is called, it is added to the previous total. By using SOGOL as an aid to addition, students understand the pattern required to reach the target number and hence the rule to win. (Students may write the numbers on the strips using sketch pens; this can be erased later). The game can also be played by subtracting numbers starting from 20, with the winner being the one who reaches 0 first.



Figure 4

### Hiding Dots

To introduce subtraction, show 9 (initially take 5 or 4) dots and ask them to count the dots; then hide some dots by folding without showing how many are hidden. Now ask how many dots are hidden. Two children can play, or they can play with parents at home.

Hold the strip in the fist and leave some of the part of the strip out. Show it to the children for 3 seconds and ask them to estimate the number of dots out of the fist and then the number of dots in the fist.

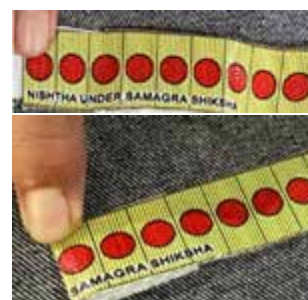


Figure 5

### SOGOL for Fractions

We usually represent a given fraction say,  $\frac{1}{4}$  by colouring 1 box out of 4 equal boxes. In the Sogol strip, try to visualise fraction using any two-colour dots. For example, 1 blue dot out of 4 dots  $\frac{1}{4}$ ; 3 orange dots out of 4 dots  $\frac{3}{4}$ .



Figure 6

Think, what are the other ways to represent the above fractions on the Sogol strip?

5 orange dots out of 8 dots  $\frac{5}{8}$

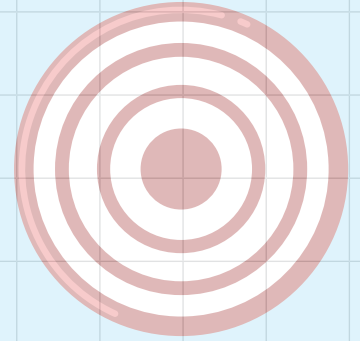
Also, 3 blue dots out of 8 dots  $\frac{3}{8}$



Figure 7: Fractions using two colours

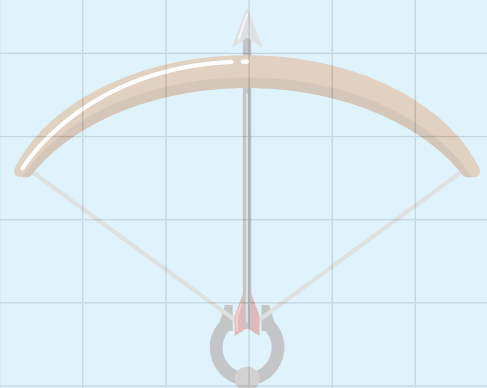
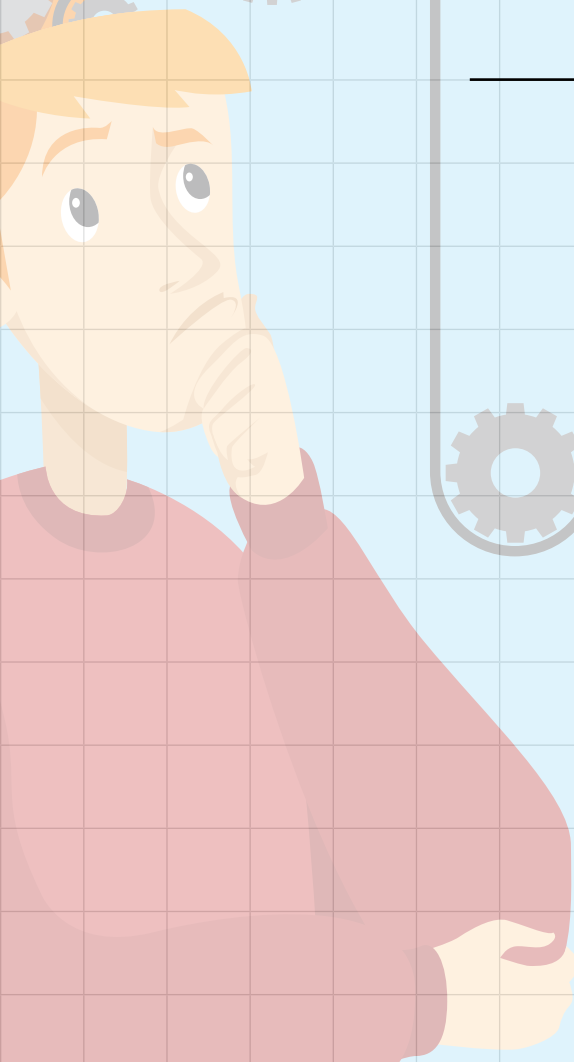
These are just a few of the ways that SOGOL can be used. For more details, contact Sunil Bajaj [bajajsunil68@gmail.com](mailto:bajajsunil68@gmail.com)

**At  
Right  
Angles**  
A Resource for School Mathematics



# LOGIC, REASONING AND PROOF

PADMAPRIYA SHIRALI



**Azim Premji  
University**

A publication of Azim Premji University  
together with Community Mathematics Centre,  
Rishi Valley

# LOGIC, REASONING AND PROOF

The first steps to proof, which is formally introduced in high school, are logic and reasoning. The school curriculum and the problem selection right from primary grades needs to bring about this aspect in a conscious manner. The curriculum should provide for questions that require justification.

At the primary level logical thinking happens in relation to concrete events and experiences. Children make logical connections through experience, knowledge, interactions, and form conclusions. There is a process of trial and error through which they internalise, arrive at conceptual understanding and logical thinking begins to take shape. As children get older, 10 years and above, they begin to think logically about abstract concepts.

Problem solving, in general, does make use of logic and it is difficult to classify problems under the category of logic. However, some problems can be identified as those that require computational skills, procedural knowledge and that are dependent on memory and conceptual clarity. Some problems require application of logic and an ability to see connections and bring forth a deeper understanding of the properties or relationships that exist between numbers or shapes. They enhance the ability of the students to think through problems and apply strategies for solving them. They build the ability to make a conjecture or an educated guess and to prove it. Students also begin to appreciate how to show proofs without words.

Exposure to problems of this nature will slowly build the skills of reasoning, justification and leads to clarification of concepts. Problems of this nature can be worked on either singly or in pairs to be followed up by discussion/ presentation to the rest of the class.

Naturally, in this process, the first stage for students is to feel completely convinced by the logical thought process that they have used. The second stage is to be able to communicate their thought processes to another student. The third stage is to be able to either satisfactorily answer any questions that are raised or a challenge posed to the thinking process that has been used.

In a classroom, giving scope for raising and inviting questions, encouraging students to make predictions, explore and observe patterns, and make connections will enhance logical thinking. As we give more and more opportunities for children to articulate their reasoning, the level of the engagement with which students approach problems increases. There is a greater likelihood of developing a sense of confidence and greater clarity.

There is good scope for providing such questions in all areas of mathematics and at all levels. The complexity of such problems can be increased gradually. In the initial stages, the problems may require a single step reasoning, later two step reasoning and so on. Students may also employ different modes of representation to prove their point. Some may use drawings, some writing, some symbols. Through the process of sharing, they learn to refine their thinking and presentation methods, and notice the flaws in their arguments. It helps students to develop a greater level of skepticism.

An investigation leading to a discovery followed by a proof can be highly satisfying for any budding mathematician.

**Keywords:** Proof, questioning, logic, reasoning, justification, conjecture, number, shape, pattern

# *“To Prove something is to Demonstrate its Truth.”*

Students should have exposure to informal ways of presenting their proofs in many contexts before being exposed to formal proof writing. Proof requires sequential reasoning which is to be built gradually. Logical thought process must be firmly established before being required to write and aim for rigour in writing. Understanding and proof evolve together.

The process of proof is a movement from selection of some properties of a given fact (premise) to arrive at a conclusion. The given information may have multiple facets and depending upon the facet selected one may arrive at a particular conclusion.

What is a proof? It is a logical argument that establishes the truth of a statement. What is logic in turn? Is it a series of steps where each is derived from the earlier steps? The process involves deduction. In real life we use logical thinking at various points of time to make decisions. While facing a problem we use the available facts to solve the problem.

For something to be called a proof, is it adequate if a statement works for a particular situation? Is it adequate if it works for many situations? Also, to verify the truth of something students need to be exposed to statements which are true under certain given conditions and learn to qualify their statements.

How do children prove something? They may give examples as proof. While an example can work as a demonstration of a principle, it does not suffice as a proof. Here is an opportunity for the teacher to show that an example-based justification is not adequate as a proof. Students may show something through the form of drawing which is a component of mathematical thinking. Experimentation precedes the reasoning process and often one starts with that step before moving into theoretical explanation.

It is necessary to restate the explanation given by the students in the correct form. It is important also to note the information that is assumed as true.

Time does often become a constraint to persist in the process due to the demands of a heavy curriculum but the time spent on building reasoning brings about a deepening of mathematical thinking.

Research has shown that students are able to appreciate proof even if they are unable to produce it themselves. The implication of this is that exposure to arguments that are meaningful can help them to slowly build skills of reasoning.

Here are some suggestions for introducing proof into the classroom at various levels. Some involve numbers, some geometry, while others involve combinations and graphs.

As one can see, the process of asking why and learning to justify can be developed right from an early stage.



Figure 1

## PROBLEM 1

If  $X + Y = X + Z$ , then can you show that  $Y = Z$ ?

When I posed this question in Grade 4, many students used numbers and drawings to show me why it is true. A few tried to give different values to  $Y$  and  $Z$  as they were different letters and found that the sum on the two sides were not equal. This required pointers from my side so that they could understand that equality is a given condition for them to work with and see its implication on  $Y$  and  $Z$ . One student came up with a unique solution by bringing a book and a pencil. He placed them on one side and placed another book and a similar pencil on the other side and said if  $Y$  is a pencil,  $Z$  will need to be a pencil for them to be equal.

It led to an interesting discussion as some students said that the other pencil was a little shorter and hence, they were not equal.

We ended up talking about when do we say things are equal, when do we say things are the same, and when do we say things are similar.



Figure 2

## PROBLEM 2

Prove that the sum of two even numbers is always an even number.

Most children attempted this problem by taking two even numbers and demonstrating by example that their sum is even. I did not object to this but prodded them to do it through a drawing. I had introduced even and odd numbers in the earlier year through a pairing activity.

As they made dot drawings to represent the numbers, a few who could recollect the previous process started to circle the pairs in the drawings. Suddenly someone noticed that in each set there was no dot left unpaired and hence both the sets when brought together had to be even. The student was able to convince the others of the logic as it was presented pictorially.

In no time this discovery led to extensions of the same logic to prove two other results.

- Prove that the sum of two odd numbers is always an even number.
- Prove that the sum of one odd and one even number is always an odd number.

Once the idea took root most students used drawings and the idea of pairing to prove the results.

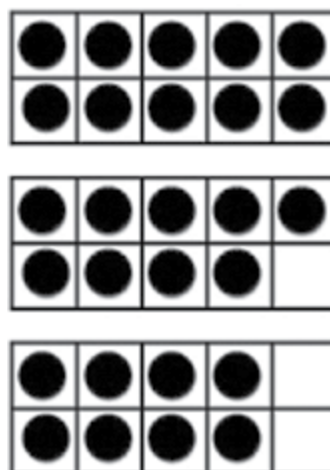


Figure 3

Students began to ask what would happen if we added three odd or even numbers. Would the answer be odd or even?

It was a good demonstration to me on how a few students' learning can affect others and the group can move forward together!

## PROBLEM 3: EXPLORATION OF CONSECUTIVE NUMBERS IN GRADE 5

Prove that the sum of three consecutive numbers is a multiple of 3.

1,2,3	2,3,4	3,4,5	4,5,6	5,6,7	6,7,8
-------	-------	-------	-------	-------	-------

We started this problem as an exploration of consecutive numbers and not with the statement given above. Students listed sets of three consecutive numbers. What can we do with these numbers? What happens if we add them? What will happen if we multiply them?

We first summed the numbers and listed the results.

6	9	12	15	18	21
---	---	----	----	----	----

Students noticed that the sums were all multiples of 3. They tried this with a few more numbers and saw that the pattern continued.

Why does it happen that the sums are all multiples of 3?

When a question arises out of a discovery, it acquires greater interest as it is not a problem posed by someone else for the student to resolve. They feel an ownership over the problem.

A few attempts were made to explain but were not very satisfactory. I needed to interject and asked them about what they noticed about each set. Various responses came up, and we took note of the fact that each set has one number which is a multiple of 3. In fact, that itself got posed as a question. *If you have a set of three consecutive numbers, will it always have a multiple of 3? Why?*

Then I asked them what is the relationship of the numbers to one another. Again, various statements came up. The middle number is one more than the one to its left and one less than the one to its right. Another student expressed it as the one on the left is one less and two less than the other numbers. Somebody else said the one on the right is two more than the left most number and one more than the middle number.

One student said that one was a multiple of 3 and the other two together were 3 more than the first.



Figure 4

This statement caused some confusion as the way the student expressed his understanding was not correct. He was referring to the difference of the numbers summing up to 3. The explanation needed to be clarified.

I asked them if they could write each number as a multiple of 3 plus the extra.

We rewrote the sets 3, 4, 5 and 4, 5, 6 and 8, 9, 10 as

$$\begin{aligned} &3 \times 1, (3 \times 1) + 1, (3 \times 1) + 2 \\ &(3 \times 1) + 1, (3 \times 1) + 2, (3 \times 2) \\ &(3 \times 2) + 2, 3 \times 3, (3 \times 3) + 1 \end{aligned}$$

Now one student remarked that one number is a multiple of 3, another number is a multiple of 3 and 1 extra, and the remaining number is a multiple of 3 and 2 extra. So, the extras add up to a multiple of 3.

We also drew dot pictures to show the same results.

While students may manage to reasonably justify a result, it is necessary for the teacher to reframe it in precise language.

Very often each proof leads to another result to be proved.

The next challenge was to prove that the sum of three consecutive numbers is thrice the middle number.

There was a lot to discover in sets of consecutive triads!

## PROBLEM 4

Prove that the sum of two consecutive odd numbers is always a multiple of 4.

This was also attempted in Grade 5. We first listed some consecutive odd number pairs.

1, 3, 5, 7, 9  
15, 17, 19, 21  
29, 31, 33, 35

Figure 5

1 + 3	3 + 5	5 + 7	7 + 9	9 + 11	11 + 13
-------	-------	-------	-------	--------	---------

Initially they tested various pairs to check if it were so. Having satisfied themselves that the statement was true for several examples, the challenge was to figure out the reasoning behind it.

I encouraged them to start with sets which had numbers bigger than 4 and depict the number pair with sets of dot drawings. With the experience that the students had gained with earlier problems, they began to look at the numbers as a multiple of 4 and an extra. That

helped them notice the relationships between the numbers.

They first wrote 5, 7 and 7, 9 and 9, 11 as  $4 + 1$ ,  $4 + 3$  and  $4 + 3$ ,  $4 + 4 + 1$  and  $4 + 4 + 1$ ,  $4 + 4 + 3$ , and then:

$$4 \times 1 + 1, 4 \times 1 + 3$$

$$4 \times 1 + 3, 4 \times 2 + 1$$

$$4 \times 2 + 1, 4 \times 2 + 3$$

Eventually they verbalised their understanding as 'each number is a multiple of 4 and some extra'. The extras in the pair add up to 4. So, the sum of two consecutive odd numbers is a multiple of 4.

Later we split the sets 1, 3 and 3, 5 as follows:

$4 \times 0 + 1, 4 \times 0 + 3$	$4 \times 0 + 3, 4 \times 1 + 1$
----------------------------------	----------------------------------

The students could see that the situation was the same here.

## PROBLEM 5

Building exposure to proof and reasoning need not always be through a big result. It can be done by posing statements which are true and false so that students learn to identify non-true statements and begin to give justification for why something is true or false.

Are these True or False? Can you give reasons why they are True or False?

The word 'numbers' here refers to natural numbers.

- When you multiply a number by an odd number, the answer is always odd.
- When you multiply a number by an even number, the answer is always even.
- Doubling a number results in an even number.

- The sum of four even numbers is a multiple of four.
- When you multiply a number by itself, the answer is even.
- Adding three consecutive numbers results in an even number.



Figure 6

## PROBLEM 6

Students should also learn to see when something is always true, when it is true for certain situations and when something is totally untrue. This can help in paving the way for the future when they come across if and only if situations, conditional statements in proofs.

The teacher can also give examples of some statements that are sometimes true. Ex. When you add two numbers you get the same result as when you multiply them. This statement is true when both numbers are zero, or both numbers are 2. But it is not true when the numbers are 2 and 3, for instance.

Are these statements always true, sometimes true, or never true?

- If a number is a multiple of 10, it is also a multiple of 5.
- If a number is a multiple of 4, it is also a multiple of 8.

- If a number is a multiple of 9, it is also a multiple of 2.
- Adding two consecutive multiples of 5 will give a multiple of 10.
- Adding 5 consecutive multiples of 2 will give a multiple of 10.

It is also necessary to look at vice versa.

- A square is a special rectangle. Is a rectangle a special square?



Figure 7

## PROBLEM 7

Proof problems can also be posed from geometry and shapes. Here is a problem I tried with Grade 6.

What is the minimum number of faces that a 3D-shape with all plane faces (a polyhedron) can have?

When I posed this question to the students, most of them thought of a cube or cuboid shape, and confidently declared that 3D shapes will have a minimum of 6 faces. I asked if it were so.

More probing brought out other 3D-shapes that have fewer faces than 6, like pyramids and tetrahedral packs ('tetrapaks').

The question now was how can you show that all 3D-shapes have at least four faces?

This was the first time that students were encountering the usage of the phrase 'at least' and it needed some explanation.

We had to do a practical study of various shapes before we could think about the question without reference to a shape. I think it is important to let students refer to their concrete experience before

they can abstract out the essentials to visualise it on paper.



Figure 8

We then looked at the minimum number of points that were needed to make a 2D-shape. That was simple as students were aware of triangles; they said 'three.' The question then was what is needed to create the third dimension? Would one point suffice?

If we make a triangle and use the minimum number of points to create a 3D-shape, how many faces would such a shape have?

Soon everyone could see the justification for the statement that 'a 3D-shape will have a minimum number of 4 faces.'

## PROBLEM 8

**“When you cut off a piece of a shape, you reduce both its area and its perimeter.”**  
**Is this always true, sometimes true, or never true? Is it true only under certain conditions?**

Again, students can experiment with cutting paper shapes if necessary. Our intention is to focus on arriving at understanding through logic eventually but experimentation may precede the reasoning process.



Figure 9

When I posed this question to students in Grade 6, they were initially quite certain that both the area and perimeter would reduce. Their logic for area was that if a shape occupied a certain space earlier and if the shape was cut and hence made smaller it should occupy less space. This made sense.

But they also argued that a smaller shape should have a smaller perimeter. I asked if it was so?

That began to create some doubt in them and made them wonder about the truth of their statement.

We took out some paper shapes and cut them to test their perimeter lengths. We tried various kinds of cuts, zigzag cuts, curved cuts, and stepped cuts.

Slowly we moved on to the question of an exploration of what happens to a straight line when it begins to get replaced with a zigzag line or a wavy line. We also ended up with a discussion on the shortest distance between two points.

It became evident that cutting off a piece can affect the perimeter in three different ways. The perimeter can stay the same (fig 10b), perimeter can reduce (fig 10a) and perimeter can also increase (fig 10c) depending upon the nature of the cut.



Fig 10a

Fig 10b

Fig 10c

This was a demonstration to me on how a simple question can lead to a lot of exploration.

What happens if the paper is folded as a rectangle and an L-cut is made in the middle on the fold?

## PROBLEM 9

Here is a problem which can be tried with students of Grade 6 or 7.

Is the statement “the value of double the number is always greater than the number” true? Justify your answer.

At the outset the statement may seem true to many students, but once they begin to look at different types of numbers, they will see where it does not always hold true.



Figure 11

## PROBLEM 10

**“If a number is divisible by 10 and is also divisible by 15, then it is divisible by 150.”  
Is this statement true? Justify your answer.**

While solving proof problems students need to bring to the fore their knowledge of concepts and facts that have been learnt earlier. In the given problem students will use their knowledge of factors, prime factorisation and lowest common multiple in reasoning out the falsity of the statement.



Figure 12

## PROBLEM 11

**Prove that the product of two consecutive whole numbers is always even.**

Students will make use of their knowledge of multiplication as repeated addition in providing their reasoning for this statement.

## PROBLEM 12

Here is a problem which can be tried at Grade 6, 7 level.

**Prove that when  $b$  is a positive integer, the value of  $3b$  is always a factor of the value of  $12b$ .**

Students will use their knowledge of positive numbers, factor-multiple relationships to establish the truth of this statement.

A further exploration can occur whether this statement holds for negative integers.

The teacher may like to extend it further and introduce the idea of generalisation by replacing 3 by  $k$  and 12 by  $nk$ .



Figure 13

## PROBLEM 13

By posing problems which involve a good understanding of number properties and laws of arithmetic students are challenged to delve deeper into their grasp of the subject matter.

Prove that the difference of two multiples of 3 is itself a multiple of 3.

What facts and laws of arithmetic will students apply in order to reason out their answer to this poser?

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

Figure 14

## PROBLEM 14

Here is a problem which I have tried at Grade 7, 8 level.

Prove that when one is added to the product of two consecutive positive even numbers, the result is a square number.

I have found it useful to urge students to use dot arrays in trying to find a proof to this problem. As they juggle around with the dot array and

rearrange the dots, they begin to perceive how the rearrangement leaves behind one missing dot to form a square array.

Students who are comfortable with algebraic manipulation may find an algebraic proof for this problem.

The teacher can link the dot arrangement and the algebraic proof.

## PROBLEM 15

Problem involving permutations and combinations:

How many different 3-digit numbers can you make using 1, 2 and 3? Can you justify that as the maximum number?

These problems make use of systematic ordering strategies as a way of approaching solutions. Students begin with one number and work out the possible combinations before moving onto the next.



Figure 15

## PROBLEM 16

Encourage students to become comfortable with algebraic expressions and defend their understanding.

Which is bigger,  $3n$  or  $n + 3$ ? How will you justify your answer?

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## PROBLEM 17

Here is a problem which can be tried at Grade 8 level.

Three numbers  $m$ ,  $n$ ,  $p$  have the property that  $m$  divides  $n$ ,  $n$  divides  $p$ , and  $p$  divides  $m$ . What must be true about these numbers? Prove your conjecture.



Figure 16

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## PROBLEM 18

Another problem at Grade 8 level.

Given two distinct numbers,  $X$  and  $Y$ , prove that their mean  $(X + Y)/2$  lies between  $X$  and  $Y$ .

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## PROBLEM 19

A problem which requires a good understanding of the multiplication process and associated logic.

What is the largest digit in the product  $11111111 \times 11111111$ ? How can you be sure you have the right answer?

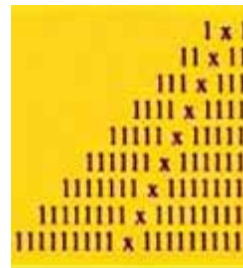


Figure 17

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## PROBLEM 20

Another problem involving permutations. Permutations and combinations are a topic in which problems involving reasoning and supporting arguments are easy to find.

How many three-digit numbers containing only even digits are divisible by 9? Can you justify your answer?



Figure 18

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## PROBLEM 21

Here is one for Grade 7 involving the concept of rounding.

When a certain four-digit number is rounded to two significant figures, the answer is 8000. What is the greatest value the number could be? What is the smallest value the number could be?

Justify your answer.

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## PROBLEM 22

Here are a few problems for middle school students for group work.

Credit: <https://www.youcubed.org/tasks/paper-folding/>

Start with a square sheet of paper and make folds to construct a new shape. Explain how you know the shape you constructed has the specified area.

- Construct a triangle with exactly  $\frac{1}{4}$  the area of the original square. Convince your partner that it has  $\frac{1}{4}$  of the area.
- Construct another triangle, also with  $\frac{1}{4}$  the area, that is not congruent to the first one you constructed. Convince your partner that it has  $\frac{1}{4}$  of the area.
- Construct a square with exactly  $\frac{1}{2}$  the area

of the original square. Convince your partner that it is a square and has  $\frac{1}{2}$  of the area.

- Construct another square, also with  $\frac{1}{2}$  the area, that is oriented differently from the one you constructed in task 3. Convince your partner that it has  $\frac{1}{2}$  of the area.



Figure 19

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## PROBLEM 23

Credit: NRICH



Figure 20



Figure 21

There are four sticks (two sets of parallel sticks) which make four crossings.

How many crossings do five sticks (with two sets of parallel sticks) make?

Still keeping two sets of parallel sticks, this time with seven sticks in total, can you arrange them in another way, to get a different number of crossings?

- What is the least number of crossings you can make?
- What is the greatest number of crossings you can make?
- Can you find all possible numbers of crossings with seven sticks?

What do you need to do to prove that you have them all or how could you show that you have them all?

## PROBLEM 24

Here is a problem for Grade 8.

In problems that are posed it is important to select the appropriate ones and to be clear of the ideas that one is trying to communicate and whether students can attempt the questions. If they are too difficult and students cannot make any headway into them, then the purpose is lost.

The lengths of the sides of a right-angled triangle are all integers. Prove that if the lengths of the two shortest sides are even, then the length of the third side must also be even.

Most students use algebra and their knowledge of Pythagoras theorem to solve this problem. Is there a geometric solution to this problem?



Figure 22

## PROBLEM 25

Here is a question based on a proper understanding of distance-time graphs.

“If a person walked around in a circle around his home, the time-distance graph would be like a circle.” Is this statement true or false? Justify your answer.



Figure 23

## PROBLEM 26

Show that if you add 1 to the product of four consecutive numbers, the answer is always a perfect square.

We experimented with some sets of four consecutive numbers.

1,2,3,4	2,3,4,5	6,7,8,9	11,12,13,14
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The product of 1,2,3,4 is 24, and  $24 + 1 = 5^2$ .

The product of 2,3,4,5 is 120, and  $120 + 1 = 121 = 11^2$ .

The product of 6,7,8,9 is 3024, and  $3024 + 1 = 3025 = 55^2$ .

The challenge was to find the proof. We tried the algebraic approach and found ourselves stuck with complicated expressions like  $a^4 + 6a^3 + 11a^2 + 6a + 1$  which was not helpful.

We looked again at the numbers arising in the problem and noticed a very distinctive pattern



Figure 24

when we multiplied the two numbers at the extremes and the two numbers in the middle:

- In the case of 1,2,3,4, we noticed that  $1 \times 4 = 4$ ,  $2 \times 3 = 6$ , and 5 is between 4 and 6.
- In the case of 2,3,4,5, we noticed that  $2 \times 5 = 10$ ,  $3 \times 4 = 12$ , and 11 is between 10 and 12.
- In the case of 6,7,8,9, we noticed that  $6 \times 9 = 54$ ,  $7 \times 8 = 56$ , and 55 is between 54 and 56.

In each case, the two products obtained were seen to be consecutive even numbers.

We now experimented with dot drawings and rearrangements to slowly unravel the proof. Here is how it worked out in the case of 1,2,3,4. We first constructed the dot diagram for  $4 \times 4$ :

o o o o o  
o o o o o  
o o o o o  
o o o o o

Removing the last column, converting it into a row, and then placing it at the bottom of the diagram, we obtained the following:

o o o o  
o o o o  
o o o o  
o o o o  
o o o o

We immediately saw that we have a square array with exactly one missing unit. We now understood why  $(4 \times 6) + 1$  is a square:  $(4 \times 6) + 1 = 5^2$ .

We then worked with another set 2,3,4,5 and tried the same approach:  $2 \times 5 = 10$ , and  $3 \times 4 = 12$ .

Here are the pictures for the two cases, using a grid rather than a dot array:

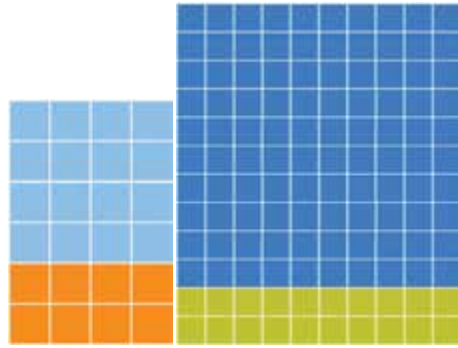


Figure 25

(Credit for the two pictures: Swati Sircar)

We realised that this process will work for the product of any four consecutive numbers.

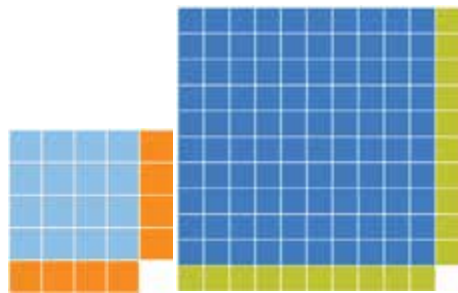


Figure 26

## PROBLEM 27

Here is a problem for Grade 8.

Here are two visual aids to help students to prove the given results. Students need to articulate their interpretation of the drawings. Teachers may need to ask some leading questions to start the students in reading from the diagram. They will then need to use their knowledge of area and algebra to arrive at the proof.

How does this diagram show that the sum of a positive number and its reciprocal is at least 2?

Credit: (Nelson, p. 62)

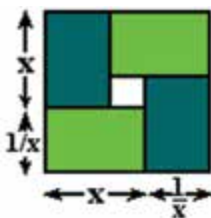


Figure 27

How do the two pictures below “prove” that  $a^2 + b^2 \geq 2ab$ ?

Students will need to notice the difference between the two pictures to interpret the way the reconstruction has happened and use their knowledge of area and deductive reasoning to arrive at the proof.



Figure 28

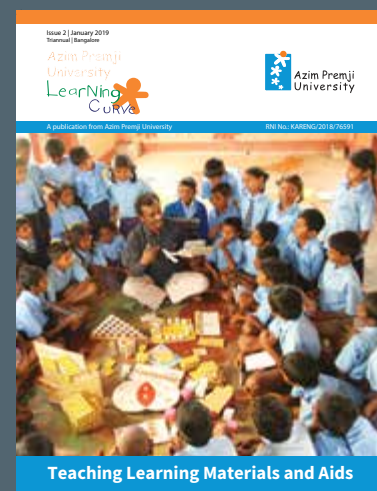
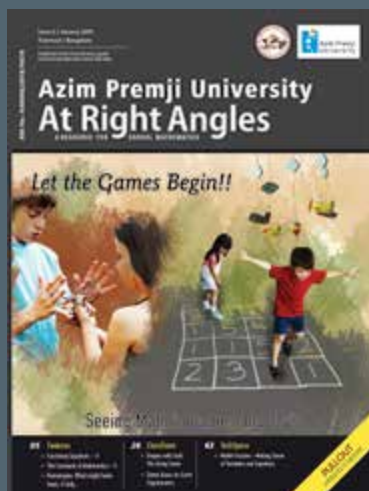
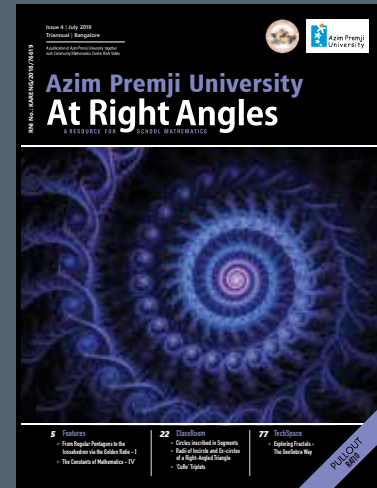
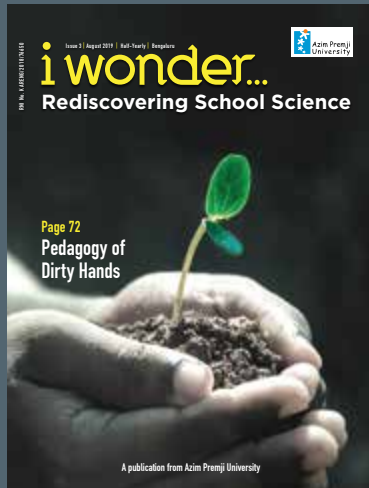




PADMAPRIYA SHIRALI

PADMAPRIYA SHIRALI is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. In the 1990s, she worked closely with the late Shri P K Srinivasan. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box.' Padmapriya may be contacted at [padmapriya.shirali@gmail.com](mailto:padmapriya.shirali@gmail.com).

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