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Azim Premji University

Azim Premji University
At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

ISSN 2582-1873

A mathematician reads between the lines . . .

The ability to see patterns and appreciate the elegance and aesthetics of mathematical concepts and ideas.

Focus on the scientific exploration of concrete experiences of the students.

Multidisciplinary capacities across languages, Mathematics, Sciences, Social Sciences, Vocational Education, and Art.

Eliminating the fear of Mathematics that is widely prevalent today.

Students also explore the relationship between Science, Technology, and Society.

PULLOUT
NEWSPAPERS IN THE MATHEMATICS CLASS

Mathematics is a very literal subject, isn't it? It spells it out like it is and proves all that it holds true! But can a mathematician take everything at face value? Should students of mathematics be trained never to contradict all that the teacher and the textbook say? How will the study of mathematics bring rigor? Only by careful examination, reading between the lines, weighing the implication of every statement and testing every hypothesis against personal experience. Good life skills wouldn't you say?

From the Editor’s Desk . . .

With the November issue comes a sense of closure- the year which started with the usual hopes and expectations has played itself out in reality and while the highs may not have been as high as we dreamed, the lows were definitely not as low as we feared! At Right Angles has had a long run in its present avatar and as the world grows ‘softer’, we are moving more and more online. You, our readers, would have noticed that we have been featuring many of our articles only in the online version and 2024 will continue and sharpen this trend. Let me explain.

Azim Premji Foundation works to improve the school education system in India, with a focus on the more disadvantaged areas of the country. In order to be a good-quality learning resource for practicing teachers, teacher educators and educational functionaries of school education, the articles in the print edition will be focused on helping to build teacher capacity. They will be directed towards facilitating more experiential and meaningful teaching-learning processes inside classrooms and to support the engagement of the school with the communities that they serve. While we have carried several articles which work to this end in past issues, the print version (which will also be available online) will now focus on such articles with emphasis on laying a solid foundation in mathematics- which means more mathematics pedagogy articles at the primary and upper primary level.

At the same time, we have always encouraged explorations and problem solving in mathematics and have built up an active community of readers and contributors. This momentum should continue, particularly the encouraging trend of more articles coming in from student contributors - and we are exploring ways in which this can happen.

And here are the highlights of the November 2023 issue! We feature the second part of Shailesh Shirali’s article *Two New Proofs of the Pythagorean Theorem* – the charm is in the journey as you will see. We have three authors contributing articles on patterns in numbers: Meera Bhide harnesses these to *Compute Squares of Consecutive Numbers,* Sujatha Singha visualizes the *Sums of Powers of Any Composite Number* and Hara Gopal defines *Haras Numbers.*

Recreation and Mathematics- an unlikely pair for most people, excluding the readers of At Right Angles - and in the November issue Shyam Sunder Gupta makes *Amazing Shapes using Factorial Digits,* Sreya Mukherjee designs an *Integer Board Game* and Vanshika Mittal describes her adventures with *Nesting Platonic Solids.*

Finding connections- here are three articles which do that! Akash Maurya finds *Connections between Paper Folding, Geometry and Proof,* Jyoti Nema & Poonam Aggarwal find a *Link between Three Trigonometric Identities for a Triangle* and Komal Asrani studies *Radius (Trijya) and Sine (Jya) - the Names and their Relationship.* The pedagogical aspects of teaching number names and numerals are addressed by Math Space in *How Much or Till What: When and Why?* A S Rajagopalan describes yet another advantage of problem solving – unlooked for bonus discoveries in *Two Fruits on One Stalk.*

Problem Corner and Student Corner have plenty of nail biting suspense- featuring *Nine-Point Centres, Zeller's Congruence* and the *Doomsday Algorithm* and more. Anushka Tonapi describes *Explorations on the Sierpinski Gasket Graph* in TechSpace and Divakaran D reviews *Adventures of a Mathematician by Stanislaw Ulam.*

And we close with the PullOut- the teaser is on the cover of the November issue….. Padmapriya Shirali welcomes *Newspapers in the Mathematics Class* and how!

Read on to find out!

Sneha Titus
Associate Editor

Opening Bracket . . .

In educational circles all over the country one of the major questions on people's minds is about the National Curriculum Framework for School Education (NCFSE) 2023, and about impending changes in school education. Predictably, much of the conversation is about changes coming to the examination system. But the vision described in the 600-page NCFSE document goes much beyond examination-related matters; it is a bold and ambitious attempt to restructure education itself. It is daunting to read the document in its entirety, but it is also an enriching task, because the document has been prepared with great care. Here are some striking quotes from Section 1 of the document.

I. "Education is, at its core, the achievement of valuable Knowledge, Capacities, Values, and Dispositions. Society decides the Knowledge, Capacities, Values, and Dispositions that are 'valuable' enough to be developed through education, and so they are informed by the vision that the society has for itself. Hence it is through the development of Knowledge, Capacities, Values, and Dispositions in the individual that education contributes to the realisation of the vision of a society."

II. "The purpose of the education system is to develop good human beings capable of rational thought and action, possessing compassion and empathy, courage and resilience, scientific temper, and creative imagination, with sound ethical moorings and values. It aims at producing engaged, productive, and contributing citizens for building an equitable, inclusive, and plural society as envisaged by our Constitution."

III. "The aim of education will not only be cognitive development, but also building character and creating holistic and well-rounded individuals equipped with the key 21st century skills."

IV. "Education must develop … appropriate values, dispositions, capacities, and knowledge. A curriculum, therefore, must systematically articulate what these desirable values, dispositions, capacities, and knowledge are, and how they are to be achieved through appropriate choice of content and pedagogy and other relevant elements of the education system, and present strategies for assessment to verify that they have been achieved."

V. "Effective action needs strong motivation in addition to knowledge and capacities. Our values and dispositions are the sources of that motivation. Values refer to beliefs about what is right and what is wrong, while dispositions refer to the attitudes and perceptions that form the basis for behaviour. Thus, in addition to developing knowledge and capacities, the school curriculum should deliberately choose values and dispositions that are derived from the Aims of Education and devise learning opportunities for students to acquire these values and dispositions."

These are serious statements that need to be unpacked with care and deliberation. If as mathematics teachers we are to take forward the vitally important task of Education, we will have to ponder over these matters carefully. One implication of this is that over and above the subject that we teach (in our case, mathematics), we need to address wider and deeper issues. We need to devote time, space, and energy to discuss such matters amongst ourselves. And we need to do the same with our students.

Education is an enormous task. It may demand capacities and dispositions that we do not possess at present. But we may discover those very capacities and dispositions by engaging with these questions passionately and critically.

Shailesh Shirali

Editorial Office

The Editor, Azim Premji University
Survey No. 66, Burugunte Village,
Bikkanahalli Main Road, Sarjapura,
Bengaluru – 562 125
Phone: 080-66144900
Fax: 080-66144900
Email: publications@apu.edu.in
Website: www.azimpremjiuniversity.edu.in

Publication Team

Shantha K
Programme Manager
shantha.k@azimpremjifoundation.org

Shahanaz Begum
Associate
shahanaz.begum@azimpremjifoundation.org

Print

SCPL
Bengaluru 560 062
www.scpl.net

Design

Zinc & Broccoli
enquiry@zandb.in

At Right Angles is a publication of Azim Premji University together with Community Mathematics Centre, Rishi Valley School and Sahyadri School (KFI). It aims to reach out to teachers, teacher educators, students & those who are passionate about mathematics. It provides a platform for the expression of varied opinions & perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

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Contents

Features

Our leading section has articles which are focused on mathematical content in both pure and applied mathematics. The themes vary: from little known proofs of well-known theorems to proofs without words; from the mathematics concealed in paper folding to the significance of mathematics in the world we live in; from historical perspectives to current developments in the field of mathematics. Written by practising mathematicians, the common thread is the joy of sharing discoveries and the investigative approaches leading to them.

ClassRoom

This section gives you a 'fly on the wall' classroom experience. With articles that deal with issues of pedagogy, teaching methodology and classroom teaching, it takes you to the hot seat of mathematics education. ClassRoom is meant for practising teachers and teacher educators. Articles are sometimes anecdotal; or about how to teach a topic or concept in a different way. They often take a new look at assessment or at projects; discuss how to anchor a math club or math expo; offer insights into remedial teaching etc.

Problem Corner

Student Corner

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Continue . . .

TechSpace

'This section includes articles which emphasise the use of technology for exploring and visualizing a wide range of mathematical ideas and concepts. The thrust is on presenting materials and activities which will empower the teacher to enhance instruction through technology as well as enable the student to use the possibilities offered by technology to develop mathematical thinking. The content of the section is generally based on mathematical software such as dynamic geometry software (DGS), computer algebra systems (CAS), spreadsheets, calculators as well as open source online resources. Written by practising mathematicians and teachers, the focus is on technology enabled explorations which can be easily integrated in the classroom.

Review

We are fortunate that there are excellent books available that attempt to convey the power and beauty of mathematics to a lay audience. We hope in this section to review a variety of books: classic texts in school mathematics, biographies, historical accounts of mathematics, popular expositions. We will also review books on mathematics education, how best to teach mathematics, material on recreational mathematics, interesting websites and educational software. The idea is for reviewers to open up the multidimensional world of mathematics for students and teachers, while at the same time bringing their own knowledge and understanding to bear on the theme.

PullOut

The PullOut is the part of the magazine that is aimed at the primary school teacher. It takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine in a different section of the school.

Online Articles

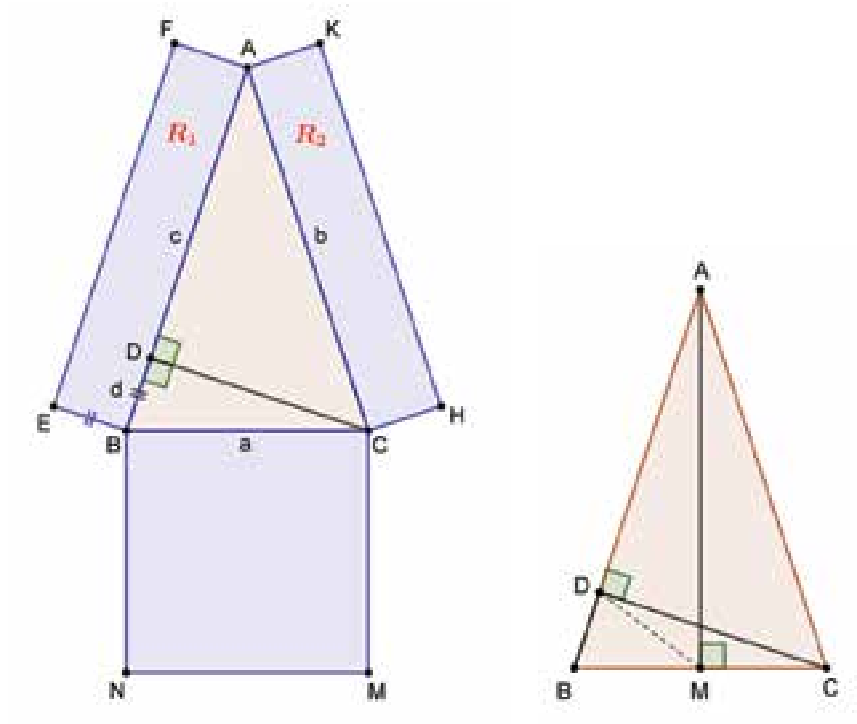
Two New Proofs of the Pythagorean Theorem - Part II

SHAILESH SHIRALI

In Part I of this article which appeared in the July 2023 issue of *At Right Angles*, we stated that "[the] Pythagorean theorem ('PT' for short) is easily the best known result in all of mathematics. What is less well-known is the fact that among all theorems in mathematics, it holds the 'world record' for the number of different proofs. There is no other theorem that even comes close! (See [2] and [3].) In the book *The Pythagorean Proposition* [1] (published in 1940), the author Elisha S. Loomis lists as many as 370 different proofs of the theorem. Since that time …more proofs have appeared." Later in the article we presented a new and very novel proof of the PT by two high-school teenagers, Calcea Johnson and Ne'Kiya Jackson, both from New Orleans, USA (see [6], [7], [8] and [9]).

Now in Part II we present an adaptation of a proof [4] by Professor Kaushik Basu, a well-known World Bank economist; he describes the proof as "new and very long" but gives a poetic and eloquent justification for adding this proof to the long list of existing proofs.

Keywords: Pythagorean theorem, Kaushik Basu, Calcea Johnson, Ne'Kiya Jackson, St. Mary's Academy, New Orleans, trigonometric proof



(a) (b)

Figure 1. The `Isosceles Lemma' and its proof

An adaptation of Kaushik Basu's "New and Very Long Proof"

As noted above, Basu describes his own proof as "very long." He starts by establishing two subsidiary results or lemmas which are of interest in themselves. While writing this article, I found that Lemma 1 ('The Isosceles Lemma') is not really needed, and Lemma 2 ('The Right-Angled Lemma') leads directly to a proof of the PT. However, for the sake of completeness, we describe both the lemmas.

Lemma 1 (The Isosceles Lemma)**.** *Let ABC be an acute-angled isosceles triangle with $AB = AC$. Draw a perpendicular CD from vertex C to side AB. Let the length of BD be d. Draw a rectangle R_1 with side AB as base and height d. Similarly, draw a rectangle R_2 with side AC as base and height d. (Of course, R_1 and R_2 are congruent to each other.) Now draw a square on side BC. Then the sum of the areas of R_1 and R_2 is equal to the area of the square.*

Proof. See Figure 1 (a). Using the language of algebra rather than geometry, we need to prove that $a^2 = cd + bd$, i.e., $a^2 = 2cd$, since $b = c$.

Drop a perpendicular AM from vertex A to the base BC; see Figure 1 (b). Since the triangle is isosceles, M lies at the midpoint of BC. Now observe that quadrilateral $ADMC$ is cyclic. This is true because $\measuredangle ADC = \measuredangle AMC$ (both are right angles). Hence we have (by the intersecting chords theorem):

$$BD \cdot BA = BM \cdot BC, \tag{1}$$

i.e., $d \cdot c = \frac{1}{2} a \cdot a$. It follows that $a^2 = 2cd$, as was to be proved. □

Remark. The Isosceles Lemma is of interest in itself, independent of its role in the proof of the PT. As we shall see, we do not really need this result to prove the PT.

Lemma 2 (The Right-Angled Lemma)**.** *Let a right-angled triangle ABC be given, with $\measuredangle B = 90°$. Locate a point D on the hypotenuse AC such that $AD = AB$. Let $CD = d$. Draw a rectangle R_3 with AC as base and height d. Draw a rectangle R_4 with AB as base and height d. Now draw a square on side BC. Then the sum of the areas of R_3 and R_4 is equal to the area of the square.*

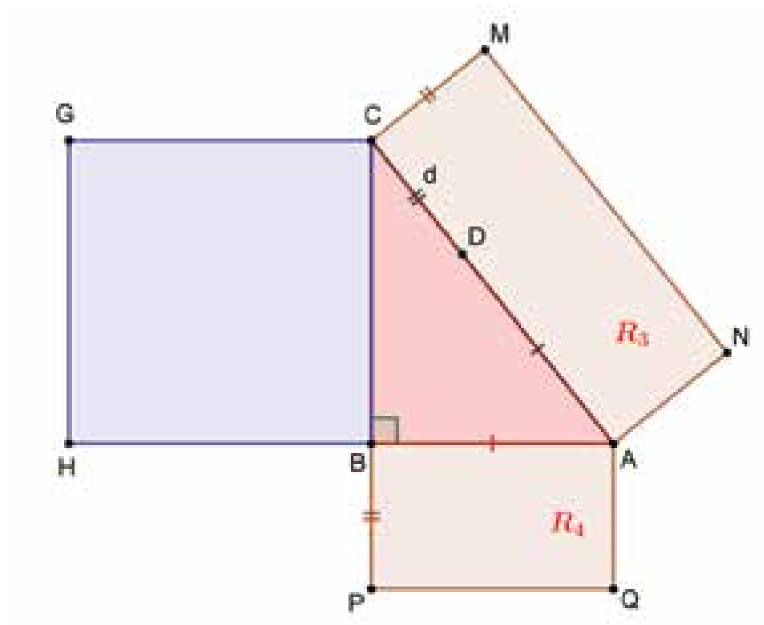


Figure 2. The Right-Angled Lemma. By construction, $AD = AB$; $CM = CD$; $BP = CD$.

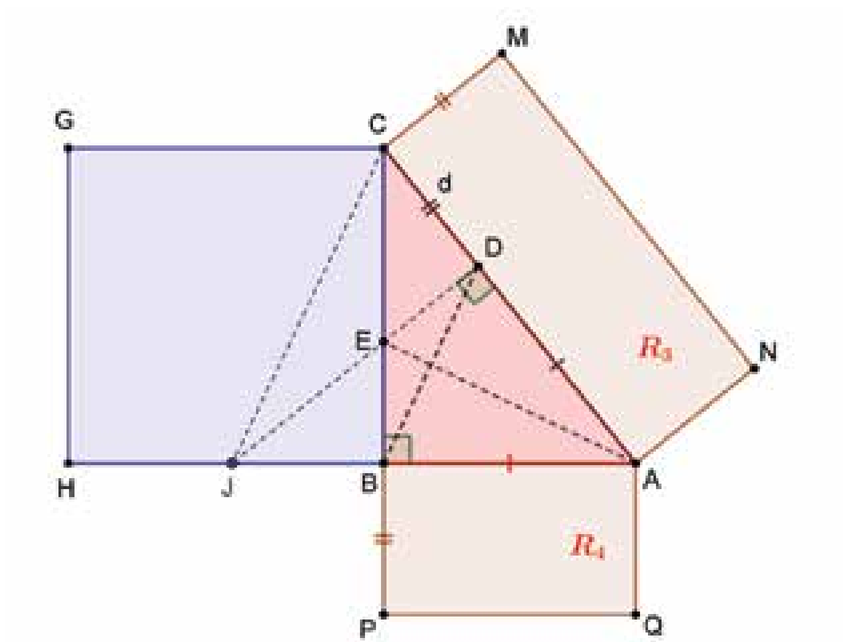


Figure 3. Proving the Right-Angled Lemma.

See Figure 2. Using the usual symbols to denote the lengths of the sides ($BC = a$, $CA = b$, $AB = c$), we must prove that $d \cdot CA + d \cdot BA = BC^2$, i.e.,

$$bd + cd = a^2. \qquad (2)$$

Proof. We draw the following additional segments in the figure: AE, which bisects $\measuredangle BAC$ and is therefore perpendicular to BD; DE; and CJ parallel to DB, with J on HB. See Figure 3.

We shall prove the stated result by showing that

$$db = CE \cdot CB, \qquad dc = BE \cdot BC. \qquad (3)$$

The stated result will then follow by adding these two relations, since $CE + BE = BC$.

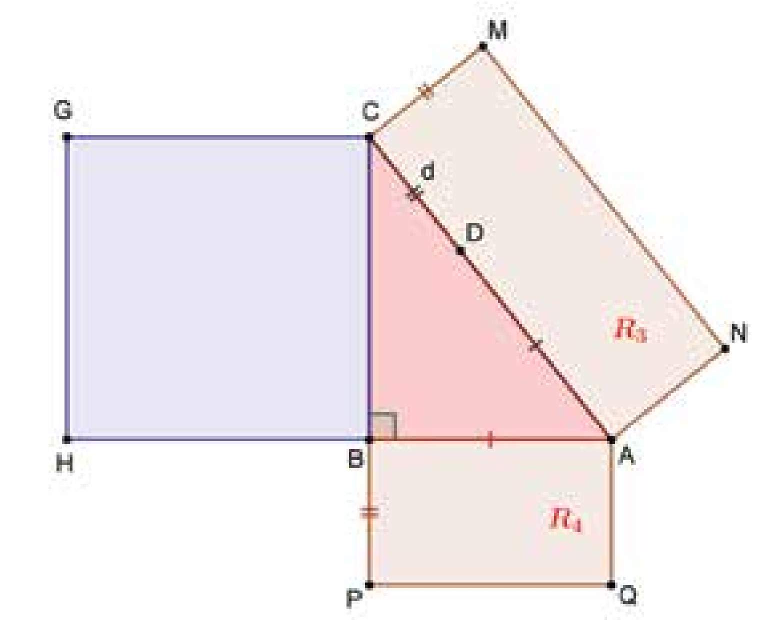


Figure 4. Proof of the PT, using the Right-Angled Lemma.

Observe that $\triangle CED \sim \triangle CAB$. Hence:

$$\frac{CD}{CE} = \frac{CB}{CA}, \qquad \therefore\ CD \cdot CA = CE \cdot CB, \qquad (4)$$

i.e., $db = CE \cdot CB$.

Next, observe that $\triangle JEB \sim \triangle CAB$. Hence:

$$\frac{EB}{JB} = \frac{AB}{CB}, \qquad \therefore\ JB \cdot AB = EB \cdot CB, \qquad (5)$$

i.e., $dc = BE \cdot BC$, since $JB = CD$ by symmetry.

So we have $db = CE \cdot CB$ and $dc = BE \cdot BC$, therefore the sum of the areas of R_3 and R_4 is equal to the area of the square, as claimed. □

Remark. As with the Isosceles Lemma, the Right-Angled Lemma is of interest in itself.

Proof of the Pythagorean theorem, based on the Right-Angled Lemma. We now show how the Right-Angled Lemma leads directly to a proof of the PT. Let $\triangle ABC$ be given, right-angled at B (see Figure 4). Using the usual symbols we must prove that $b^2 = c^2 + a^2$.

With reference to Figure 4, we have already proved that $d \cdot (b + c) = a^2$. Now by construction we have $d = b - c$. Hence:

$$\begin{aligned} (b - c) \cdot (b + c) &= a^2, \\ \therefore\ b^2 - c^2 &= a^2, \end{aligned} \qquad (6)$$

and therefore, $b^2 = c^2 + a^2$, as required. We have proved the PT! □

Remark. Basu offers the following comments to his own proof. He writes, charmingly:

> *"How then can one justify presenting a new and longer proof of Pythagoras' theorem? The only way to answer this is to invoke another Greek, Constantine Cavafy and his classic poem, Ithaca, which describes the long journey to Odysseus' home island. When you reach the island, the poet warns the reader, you are likely to be disappointed, for it will have little new to offer. But do not be disappointed, Cavafy tells the reader, for Ithaca's charm is the journey itself."*

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SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

Math is a cake walk!

Math Space @ Azim Premji University had a special work anniversary celebration recently.

We divided a delicious chocolate cake into 12 pieces and served each piece in a half plate!
Here is your challenge!
How many math questions can you make from this situation?
Send in your questions to AtRiA.editor@apu.edu.in

Computing Squares of Consecutive Numbers in a Number Series

MEERA

This article focuses on computing squares of every consecutive number in a given number series such as 10-20, 20-30, 30-40, …within a few seconds. This is done through mental calculations by following the pattern observed among the square numbers. The methodology used here is a blend of observation and trial and error methods to formulate the final working rule.

A new approach based on the pattern

The following is the special pattern observed among the square numbers. Approximately 50 iterations were carried out to identify the pattern and to develop the working rule. Only whole numbers are considered here. The last digit of the square of any number can be easily obtained by squaring the last digit of the given number. While observing the pattern in the following table, **just omit the last digit (in black font) of every square number and observe the pattern among the numbers formed by the remaining digits (in red font).**

00	100	400	900	1600
01	121	441	961	1681
04	144	484	1024	1764
09	169	529	1089	1849
16	196	576	1156	1936
25	225	625	1225	2025
36	256	676	1296	2116
49	289	729	1369	2209
64	324	784	1444	2304
81	361	841	1521	2401

Keywords: Numbers, Squares, Consecutive, Pattern

The number to be added to get the next consecutive number follows the following pattern:

00	100	400	900	1600
+0	+2	+4	+6	+8
01	121	441	961	1681
+0	+2	+4	+6	+8
04	144	484	1024	1764
+0	+2	+4	+6	+8
09	169	529	1089	1849
+1	+3	+5	+7	+9
16	196	576	1156	1936
+1	+3	+5	+7	+9
25	225	625	1225	2025
+1	+3	+5	+7	+9
36	256	676	1296	2116
+1	+3	+5	+7	+9
49	289	729	1369	2209
+2	+4	+6	+8	+10
64	324	784	1444	2304
+2	+4	+6	+8	+10
81	361	841	1521	2401
+2	+4	+6	+8	+10

Here 1 repeats 4 times, 2 repeats 6 times, 3 repeats 4 times, 4 repeats 6 times, 5 repeats 4 times, 6 repeats 6 times and the pattern continues.

Based on this, when the series like 10-20, 20-30, 30-40….are taken, the following method is used.

Working Rule: To find squares of numbers from 20-30.

Step 1: First write the square of 20.Then write the last digit of square of every consecutive number by squaring the last digit of given number.

20^2	400
21^2	1
22^2	4
23^2	9
24^2	6
25^2	5
26^2	6
27^2	9
28^2	4
29^2	1
30^2	0

Table 1. Source: Author

Step 2: Consider the lower limit of the series which is 20; omit the last digit of 20 and multiply the remaining digit by 2, which is $2 \times 2=4$.

Now add 4 to 40 (40 is taken from the square of 20 by omitting the last digit). Continue adding 4 until you get the square of number ending with 3. Then add 5 until you get the square of the number ending with 7. Add 6, until you get the square of upper limit of the series. Thus, you will get all the square numbers between 400 and 900.

Number	Square	Method
20^2	400	Omit the last digit of 20, then $2 \times 2 = 4$
	+4	
21^2	441	
	+4	
22^2	484	
	+4	
23^2	529	After getting the square of the number ending with 3, switch to next number = (4 + 1)
	+5	
24^2	576	
	+5	
25^2	625	
	+5	
26^2	676	
	+5	
27^2	729	After getting the square of number ending with 7, switch to next number = (5 + 1)
	+6	
28^2	784	
	+6	
29^2	841	
	+6	
30^2	900	

Table 2. Source: Author

Example 2: Write all the squares of the numbers from 50-60

Number	Square	Method
50^2	2500	Omit the last digit of 50, then 5 × 2 = 10
	+10	
51^2	2601	
	+10	
52^2	2704	
	+10	
53^2	2809	After getting the square of the number ending with 3, switch to next number = (10 + 1)
	+11	
54^2	2916	
	+11	
55^2	3025	
	+11	
56^2	3136	
	+11	
57^2	3249	After getting the square of the number ending with 7, switch to next number = (11 + 1)
	+12	
58^2	3364	
	+12	
59^2	3481	
	+12	
60^2	3600	

Table 3. Source: Author

Example 3: Write all the squares of numbers from 1200 to 1210

Number	Square	Method
1200^2	1440000	
	+240	Leave the last digit of 1200, then 120 × 2 = 240
1201^2	1442401	
	+240	
1202^2	1444804	
	+240	
1203^2	1447209	After getting the square of the number ending with 3, switch to next number = (240 + 1)
	+241	
1204^2	1449616	
	+241	
1205^2	1452025	
	+241	
1206^2	1454436	
	+241	After getting the square of the number ending with 7, switch to next number = (241 + 1)
1207^2	1456849	
	+242	
1208^2	1459264	
	+242	
1209^2	1461681	
	+242	
1210^2	1464100	

Table 4. Source: Author

Conclusion

This method was taught to students in the class. Students found this method very helpful as it gives the squares of entire series without performing actual multiplication. The method is easy to remember and efficiently used for all 2-digit numbers. This new approach helps to generate the squares of an entire series in a few seconds. This method improves mental ability as well as increases the pace of calculation. To generate the squares of given series of numbers this method seems amazingly easy.

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MEERA is a lecturer with nine years of teaching experience for postgraduates, undergraduates, and pre-university courses. She has coordinated workshops and seminars for the active participation of students to enhance their subject knowledge and develop an aptitude for research. She keeps her passion for teaching alive through her YouTube channel **meeramaths** (https://youtube.com/@meeramaths2998?si=5E7dtzPAAb-ub0LX) Meera may be contacted at meeranathu@gmail.com

INTERESTING NUMBERS

A deeper look at numbers reveals many interesting facts. Here are some interesting numbers.

I. The number **1210** is a very interesting number. When 1210 is divided by the number formed by its last two digits from the right (10), the quotient is the square of a number which is 1 less than its first two digits from the left.

   $1210 / 10 = 121 = 11^2$

   Can you find some other four-digit numbers which exhibit the same property?

II. The number **66** is an interesting number for two reasons.

   1. 66 can be expressed as the sum of consecutive integers using three different combinations. Here is one of them: 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 = 66

      Can you find the other two?
   2. Write down the products of 66 with 2, 3, ...10. What do you notice about the digits of the products in each case?

III. The number **365** is an interesting number. $10^2 + 11^2 + 13^2 = 365$

   Can you write **365** as the sum of the squares of two consecutive numbers?

IV. The number **11664** is an interesting number. $\frac{(1-1+6)^6}{4} = 11664$

   Can you describe this property in words and find another number which exhibits the same property?

Contributed by Wallace Jacob

Answers on page 52

Connections between Paper Folding, Geometry and Proof

AKASH MAURYA

Let us consider a rectangle ABCD. We create another rectangle ABFE within it by joining the midpoints F and E of the breadths of the original rectangle ABCD. Then the diagonal AC of the original rectangle and the diagonal BE of the second rectangle (ABFE) intersect at a point O in such a way that if we draw a straight line through O parallel to DC, which intersects the breadths AD & BC of the original rectangle at G and H respectively, then we get a third rectangle ABHG whose breadth will be **one-third (1/3)** the breadth of the original rectangle."

In general, if we apply this same process to the newly obtained rectangle, then we will get a new rectangle whose breadth will be again one third of the previous rectangle i.e., this will be a continuous process. Further we will get the same result if we get the intersection point by involving other diagonals of rectangles i.e., BD and AF respectively.

Note: This intriguing statement was made by my mentor. It happened as described below.

Keywords: Paper Folding, Geometry, Exploration, Verification, Reasoning, Proof.

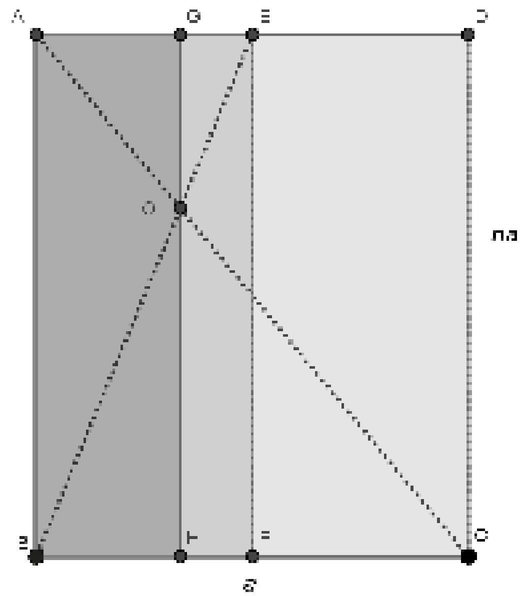


Figure 1.

One day, during my associate program, after completing my school assignments, I arrived at the District Institute. It was on that day that my mentor asked us to verify this statement through folding an A-4 size paper. Intrigued, I replicated the paper folding process with an A-4 size paper and verified its dimensions using a ruler. To my surprise, the results aligned perfectly.

This ignited a discussion between my mentor and myself, prompting him to challenge me: ”Can you prove it?” Eager to showcase my mathematical prowess and passion, I eagerly accepted the challenge. I meticulously depicted the paper-folding figure in my notebook and engaged in intense contemplation on how to formulate a proof. When I proved it, I presented it to my mentor.

Buoyed by this success, I set out to articulate the theorem and its corresponding proof in a succinct manner. This endeavour deepened my comprehension of the intricate connection between paper folding, geometry, and mathematical analysis. It became evident that geometry serves as a foundation upon which mathematical analysis can flourish, with geometric concepts often serving as a springboard for the development of intricate mathematical analyses. Here is my proof.

Proof : Let us consider the length of the rectangle (i.e., the longer side) $AB = DC = na$ unit, and breadth $BC = AD = a$ unit, where n is any real number.

Applying Pythagoras theorem in triangle ABE , we get

$$BE^2 = AB^2 + AE^2 = (na)^2 + \left(\frac{\mathrm{a}}{2}\right)^2 = (4n^2+1)\frac{\mathrm{a}^2}{4};$$

So

$$BE = \sqrt{(4n^2+1)}\frac{a}{2} \tag{1}$$

Since ΔBOH & ΔBEF are similar, we have –

$$\frac{BH}{BF} = \frac{BO}{BE} = \frac{HO}{FE} \tag{2}$$

Using $BF = \frac{BC}{2} = \frac{a}{2}$, in $\frac{BH}{BF} = \frac{BO}{BE}$, we have $\frac{2BH}{a} = \frac{BO}{BE}$

$$\Rightarrow BO = \sqrt{(4n^2+1)}BH \qquad \{\text{using equation (1)}\} \tag{3}$$

Now using equation (2) again we have –

$$\frac{BO}{BE} = \frac{HO}{FE}, \frac{2BO}{\sqrt{(4n^2+1)}} = \frac{HO}{n} \qquad \{\text{since } FE = AB = na \text{ \& using equation (1)}\}$$

which gives us,

$$HO = 2n\, BH \tag{4}$$

Similarly, ΔAOE & ΔCOB are similar, we have –

$$\frac{OE}{OB} = \frac{AE}{CB}, OB = 2OE \left(\text{since } AE = \frac{CB}{2}\right)$$

So, from Figure 1,

$$BE = 3OE = \frac{a}{2}\sqrt{(4n^2+1)} \quad OE = \frac{a}{6}\sqrt{(4n^2+1)} \quad OB = \frac{a}{3}\sqrt{(4n^2+1)} \tag{5}$$

Now using equation (4) & (5) in ΔBOH we have –

$$BO^2 = BH^2 + OH^2 = (4n^2+1)BH^2 \quad (4n^2+1)\frac{a^2}{9} = (4n^2+1)BH^2$$

Which gives us

$$BH = \frac{a}{3} \qquad \text{(Hence Proved).}$$

AKASH MAURYA has a Master's degree in Mathematics from University of Allahabad, Uttar Pradesh. He has been working as an Associate with Azim Premji Foundation since August 2022. His interests includes reading and teaching mathematics, reading editorials, listening to music, etc. Akash can be contacted on the phone at +91 9598288905 and on email at akash.maurya@azimpremjifoundation.org

Integer Board Game

SREYA MUKHERJEE

In the early years of our education, we embark on a journey of numbers and arithmetic (besides other topics), learning the basic rules and operations that govern them. When learning subtraction, we are taught that higher numbers cannot be subtracted from lower numbers, since it appears illogical. The understanding of natural and whole numbers, and to some extent, some of their basic properties, shapes our arithmetic foundation until we reach grade 6. Here, we encounter a significant leap into a whole new realm of numbers. Suddenly, we are introduced to an infinite extension of numbers in the opposite direction. The concept of integers seems somewhat counterintuitive. It is as if we have entered a whole new mathematical universe, where we must learn new rules and adapt our thinking to grasp these unfamiliar entities. This transition challenges our preconceived notions and requires us to reorient our understanding of numbers. What follows is an extensive exercise of drill and practice with worksheets full of problems on operations of integers, which is mostly monotonous and drab.

I have grown up playing "saanp-seedhi" (snakes and ladders) with my family. I remember feeling excited and proud of myself when I grew out of the need to count my position on the board by skipping over numbers one by one and could easily calculate my position in my head. My teachers however missed the opportunity of using this game (or the number chart version of this, made into a game by Jodo Gyan) as a supplementary material or an exciting context for solving addition and subtraction problems.

Gamifying drills and practice can make learning interactive and enjoyable for students, and perhaps something similar can be done with integers as well. While the NCERT textbook chapter

Keywords: Gamification, Reasoning, Integers, Number Operations

on integers in grade 6 itself suggests such a game, I have taken the liberty to gamify it a little more by incorporating opportunities to strategize and solve problems. In this article, I shall make a case for why such a game should supplement existing resources and what other avenues an integer board game can open up in a classroom.

Integer Board-Game: Rules and Game Setup

The game setup consists of the following things:

1. A board with two rectangular areas, divided by a "zero" zone in the middle. One of the zones is marked with numbers from 1 to 300 and the other zone is marked with numbers from –1 to –300. The middle "zero" zone is the starting point, where all the players must place their counters in the beginning.
2. Two blue dice signifying positive integers and two pink dice signifying negative integers.
3. Two Operations dice:
   a. One Operations die with the signs +, – marked on three faces each (Grade 6)
   b. One Operations die with the signs +, –, × marked on two faces each (Grade 7)
4. A set of wild cards and bonus cards with different instructions on them, that the teams might have to pick up on reaching certain specially marked positions on the board.
5. Counters (of different colours) to indicate position on the board.

In the beginning, the counters of all teams are placed on zero. Players then start by rolling dice picked blindly from a bag, accompanied by an Operations die. The combination of these rolls guides them in performing computations on dice numbers, which dictate the number of steps taken on the board. For example:

On the first turn, the players first draw a blue die and get 3 on rolling it, it is read as "+3". Then they draw a pink die and get 6 on the roll and read it as –6. Now they get a + sign on rolling the Operations die. Then the number of steps will be: (+3) + (–6)= –3 and thus we have to add –3 to the current position, which is zero, and thus land on –3.

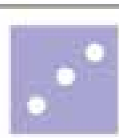
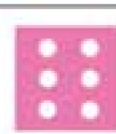
Face 1	Operation	Face 2	Number of steps
	+		– 3

Figure 1: Example of what the first turn could look like

On their next turn, if the players arrive at –10 number of steps, then their counter moves from –3 to –13. Or, if on the next turn, the players arrive at +10 number of steps, then their counter moves from –3 to the other side of zero, landing at +7.

If a special "wild" position is landed on, a wildcard is drawn and its instructions are followed. The game becomes more challenging as progress is made, with different rules for different stages. Whichever team reaches either end of the board (±300) first, wins the game.

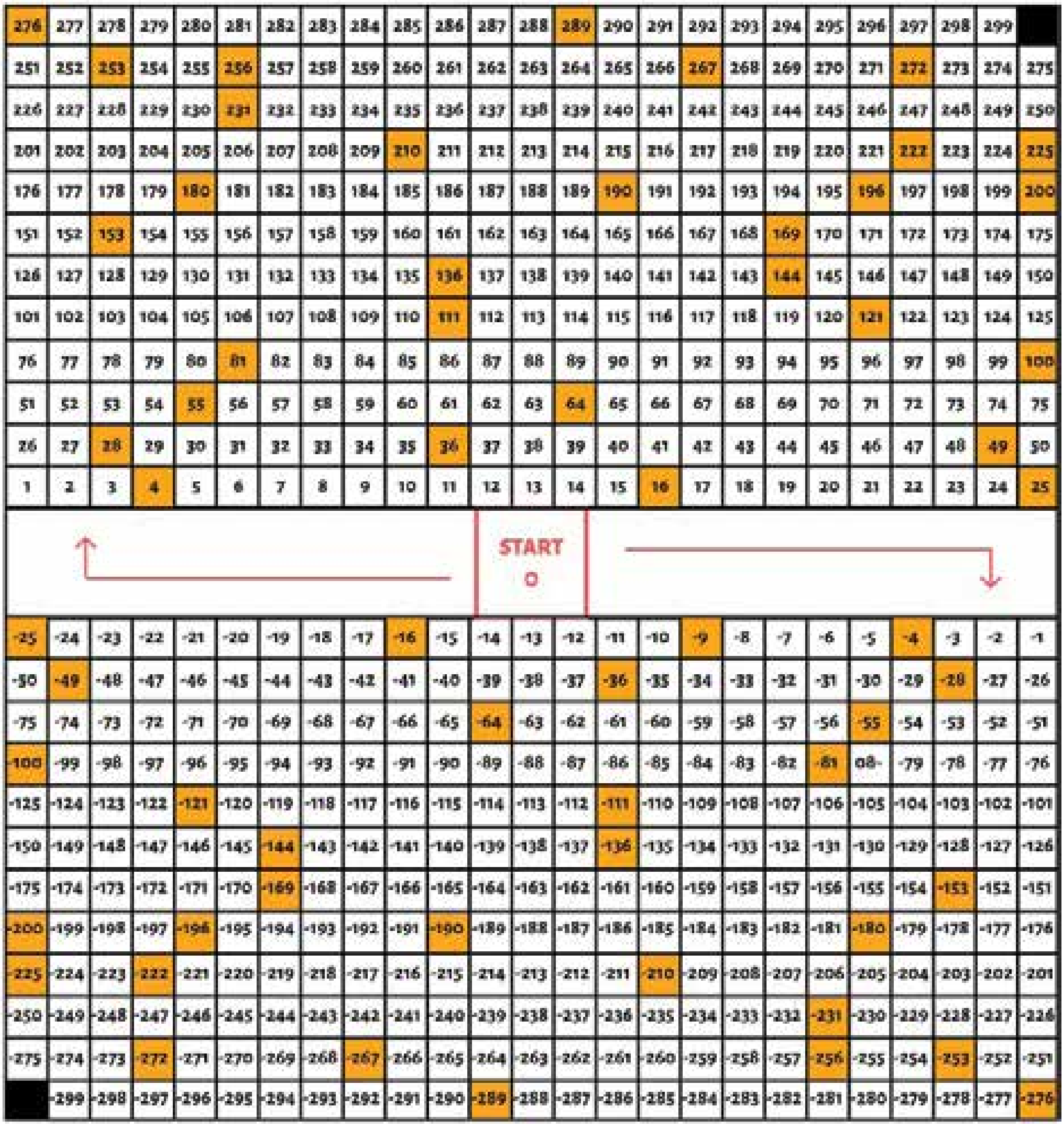


Figure 2: The game board with the full set of dice, with end points up to ±300

Progression. The operation of multiplication should be included in grade 7, after they are introduced to multiplication of integers. In both the versions of the game (grades 6 and 7+), the game can be made progressively difficult by introducing rules such as: increasing the number of times the number dice is drawn after every few turns, increasing the number of times the operations die is rolled, allowing teams to choose the number of operations, etc. The idea is to offer students opportunities to handle more calculations and take into account more considerations while strategizing.

Table 1 is a suggestive progression of complexities / levels that can be introduced in the game:

Complexity level	No. of times number dice is drawn and rolled	No. of times operations die is rolled at each turn	Operations	Description
1	2	1	+, –	Introduce basic operations. Order in which operation is applied depends on the order in which number dice were drawn. E.g.: If +3 is drawn first and –6 is drawn next, and subtraction symbol comes on the Operations die, then operating equation will be: (+3)–(–6)= +9
2	3	2	+, –	Enhance computational skills by introducing more numbers to operate and two operations to be done sequentially.
3	3	2	+, –	Strategic operation order. Let students decide the order in which numbers drawn are operated on.
4	2	1	+, –, ×	Introduce basic operations. Order in which operation is applied depends on the order in which number dice were drawn.
5	3	2	+, –, ×	Enhance computational skills by introducing more numbers to operate and two operations to be done sequentially.
6	3	2	+, –, ×	Strategic operation order. Let students decide the order in which numbers drawn are operated on.

Table 1: Complexity levels

Wild Cards

Besides adding an element of excitement to the game, the wild cards also add an exploratory flavour to the game, allowing students to solve problems, strategize and explore certain properties of operations of integers.

Figure 3: Deck of Wild Cards

These wild cards ensure that students get opportunities to operate on bigger and both positive and negative numbers, think about and put into practice multiplication facts of even negative integers, and strategize so as to minimize closeness to either end for the opponent team. A suggestive list of wild cards that can be used is given in Table 2 below.

S. No	Wild Cards	No. of cards in the deck
1	Subtract –12 from your position and place your counter there	4
2	Subtract 13 from your position and place your counter there	4
3	Subtract –25 from your position and place your counter there	4
4	Add –14 to your position and place your counter there	4
5	Add 18 to your position and place your counter there	4
6	Add –25 to your position and place your counter there	4
7	In the next turn, roll only two number dice of your choice	6
8	An extra turn! It's your turn to play again	4
9	Move to the nearest multiple of –9	1
10	Move to the nearest multiple of 10	1
11	Move to the nearest multiple of 6	1
12	Move to the nearest multiple of 4	1
13	Move to the nearest multiple of –8	1
14	Choose two number dice that the next team has to roll on their turn	4
15	In the next turn, choose your own operation	4

Table 2: A list of wild cards that can be used, along with the minimum number of such cards that should be in a 4-team game deck

While this game provides an exciting context for practicing operations on integers, it is to be noted that the game only provides one model (number line) of conceptualizing integers and assigning meaning to integer operations. Another limitation of the game is its inability to incorporate operations on large numbers (the sums or products of which might be too large) and use division as an operation on integers.

Usage in Classrooms

Concrete Representations: Due to the abstract nature of integers, especially the lack of ability to map negative integers to concrete objects, students often struggle to understand integers. The board game—an extension of a number line model representation of integers—provides students with a tangible context to understand integers and integer operations.

Application and practice of operations: The game offers an exciting setting to reinforce concepts learnt in the classroom, discover properties of integers (say, commutativity under selective operations, while strategizing on which order to perform the operations in) and a fun way to practice.

Context for assessment: The game can be used as a context for assessment questions, pushing students to both use operation rules, as well as strategies to solve problems. While the game also presents avenues for instantaneous assessment through post-game discussions and analysis of game transcripts, the game context can be extended to a more systematic formative assessment as well as summative assessment questions.

Strategizing and reflecting on strategies: At each turn, students should be encouraged to keep track of the calculations that they are carrying out. This can be done by noting down the numbers and operations that come up on the faces of the dice at each turn on a notepad or game transcript sheet. Using these notes as reference, explicit discussions on use of certain properties of numbers or strategies to one's advantage can be discussed and verified. Post-game discussions with peers and teachers—where better strategies that could have been adopted at certain points are discussed—can be used as a pedagogic tool as well.

The "Integer Board Game" offers an innovative approach to conceptualize integers and apply operations on integers and makes the concept more accessible and relatable. By providing an engaging and interactive experience for drill and practice, it supplements existing problem-solving exercises in the textbook, while providing an opportunity to learn, discuss and work together while playing a game during school hours.

SREYA MUKHERJEE is an education researcher and program designer for social impact, based out of Bangalore. She is passionate about foundational numeracy, meaningful assessments and creating a holistic school experience for students. Her other interests are cooking, reading and music. She may be contacted at sreyamukherjee1390@gmail.com.

Amazing Shapes using Factorial Digits

SHYAM SUNDER GUPTA

The factorial of a natural number n is the product of the positive integers less than or equal to n. It is written as n! and pronounced 'n factorial'. The first few factorials for n = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10... are 1, 1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800... etc. n! gives the number of ways in which n objects can be permuted. The special case 0! is defined to have value 0! = 1.

The number of digits in factorials grows very fast. For example, 6! (i.e., 720) consists of 3 digits, but the number of digits grows to 23 for 23! i.e., 25852016738884976640000. Interestingly, the digits of factorials can be represented in many amazing shapes such as triangle, rhombus, hexagon, etc., but for this, it is necessary that the number of digits in $n!$ must be such that it **can** represent that shape. In this paper, you can find as to how the factorials with required number of digits for the desired shape can be obtained.

For geometrical shapes like triangle, rhombus, hexagon, octagon, two sides are considered equal if the number of digits placed on each side is equal. So, for equilateral triangle, the number of digits of each of the three sides must be equal. The number of digits in a factorial which are required to decide/draw any shape can be computed as follows:

Keywords: Factorial, number of digits, logarithms, floor function, geometrical shapes, special numbers

The number of digits in the base 10 representation of a number x is given by

$\lfloor \log_{10} x \rfloor + 1$, where $\lfloor m \rfloor$ is the floor of m, the largest integer less than or equal to m. The log of the factorial function is easier to compute than the factorial itself. For any $n > 0$, the number of digits in $n!$ i.e. $d(n!) = \lfloor \log_{10} n! \rfloor + 1$.

For example, $d(23!) = \lfloor \log_{10} n! \rfloor + 1 = \lfloor 22.41 \rfloor + 1 = 22 + 1 = 23$. Table 1 gives several examples.

S.No.	n	Number of digits in $n!$	S.No.	n	Number of digits in $n!$
1	5	3	20	335	703
2	6	3	21	350	741
3	9	6	22	381	820
4	13	10	23	413	903
5	17	15	24	446	990
6	32	36	25	463	1035
7	38	45	26	480	1081
8	44	55	27	570	1326
9	65	91	28	589	1378
10	106	171	29	608	1431
11	125	210	30	647	1540
12	135	231	31	667	1596
13	156	276	32	687	1653
14	178	325	33	728	1770
15	201	378	34	749	1830
16	213	406	35	770	1891
17	278	561	36	880	2211
18	292	595	37	996	2556
19	306	630			

Table 1

Equilateral triangles from factorial digits

It can be seen from Figure 1 that the first row consists of 1 digit, second row of 2 digits, third row of 3 digits and so on. So, the n^{th} row consists of n digits. So, the number of digits in any triangle is the partial sum of the series $1 + 2 + 3 + 4 + 5 + \ldots n$, which is always a triangular number given by $\frac{n(n+1)}{2}$. So, if the number of digits in $n!$ is a triangular number, then the digits of that factorial can be represented in the form of triangles as shown in Figure 1. There are 37 factorials below 1000! for which the number of digits is a triangular number greater than 1 and these are shown in Table 1. It can be seen that this triangular shape is actually an equilateral triangle that has all three sides of equal length (i.e., equal number of digits).

6
22
702
0800

13!

8
24
765
0592
08247
066672
3170306
78549625
218625855
1345437492
92212313438
895577497600
0000000000000

65!

1
58
520
2313
40322
891214
0250060
00369197
521322331
5151218259
22243249182
116249442644
0719269811610
86408909404739
619639922847404
8817862008387700
58117447251615518
816153806034311423
4267647167947777284
11569911784512291060
691664690376713087265
5087823727239223545513
04572738521493300705432
412738235136532691872618
0319633667807143644833723
18720000000000000000000000
000000000000000000000000000

201!

Figure 1

Rhombus from factorial digits

It can be seen from Figure 2 that rhombus can be represented as a combination of two triangles, one with $\frac{n(n+1)}{2}$ digits placed upside down below the base of the other triangle with $\frac{(n+1)(n+2)}{2}$ digits. Since the sum of two consecutive triangular numbers is always a perfect square, if the number of digits in a factorial is a perfect square, then the digits of that factorial can be represented in the form of rhombus as shown in Figure 2. It can be seen that this rhombus shape has all four sides of equal length (i.e., equal number of digits) and two unequal diagonals. There are 20 factorials below 1000! for which the number of digits is a square number greater than 1 and these are shown in Table 2.

S.No.	n	Number of digits in $n!$	S.No.	n	Number of digits in $n!$
1	7	4	11	284	576
2	12	9	12	304	625
3	18	16	13	367	784
4	32	36	14	389	841
5	59	81	15	435	961
6	81	121	16	483	1089
7	105	169	17	508	1156
8	132	225	18	697	1681
9	228	441	19	726	1764
10	265	529	20	944	2401

Table 2

2
63
130
8369
33693
530167
21801
2160
000
00
0

32!

5
79
712
6020
74736
798587
9734231
57810910
541235724
4731625958
74586504971
6390179693
892056256
18453424
9745940
480000
00000
0000
000
00
0

81!

1
11
824
8651
19600
430744
9963076
07616902
997562475
5718426338
38412167568
361169672820
1184540457302
60688510087990
927196104962685
46259583736033
6094267205134
948250389032
46192490976
6607715924
086489297
71520000
0000000
000000
00000
0000
000
00
0

132!

Figure 2

Hexagon from factorial digits

It can be seen from Figure 3 that the first row consists of d digits (where d is the number of digits in each side), each subsequent row consists of 2 digits more than the previous row till we reach the d^{th} row. After the d^{th} row, each subsequent row consists of 2 digits less than the previous row till we reach the last row that is the $(2d-1)^{th}$ row. So, the number of digits in any such hexagonal shape is $4 \times d \times d - 5 \times d + 2$. So, if the number of digits in n factorial is equal to $4 \times d \times d - 5 \times d + 2$, then the digits of that factorial can be represented in the form of a hexagon as shown in Figure 3. As shown in Table 3, there are 18 factorials below 2000! which have $4 \times d \times d - 5 \times d + 2$ digits. It shall be noted that each side of the hexagon consists of d digits.

S.No.	n	Number of digits in $n!$	S.No.	n	Number of digits in $n!$
1	11	8	10	477	1073
2	23	23	11	527	1208
3	57	77	12	690	1661
4	78	116	13	936	2377
5	129	218	14	1142	2998
6	158	281	15	1289	3452
7	190	352	16	1444	3938
8	224	431	17	1691	4727
9	299	613	18	1955	5588

Table 3

40526
9195048
772167556
80601905432
3221349803847
96226602145
184481280
0000000
00000

57!

1020191707388
135453451234870
99089543129529601
3911923331389255804
198255110589056224885
05097140825857783313049
2962339087896214213025863
343847422809759689915199470
75158976757318201336004071925
4173922847519884523011689295750
881072987547664550681921496101296
36584647829875145351075326925507090
2741458798163733762262251227685707645
60647883318820166055483409238833395
058848823113370742011576574688777
7528611689567167232647229501179
17379184970096375469052384943
427428312650878860048411744
3856454940778122664181980
31758940202687410937462
4612936056832000000000
0000000000000000000
00000000000000000
000000000000000
0000000000000

299!

Figure 3

Octagon from factorial digits

If d is the number of digits in each side and number of digits in a factorial is equal to $7 \times d \times d - 10 \times d + 4$, then digits of that factorial can be represented in the form of an octagon as shown in figure 4. As shown in Table 4, there are 19 factorials below 5000! which have $7 \times d \times d - 10 \times d + 4$ digits. It shall be noted that each side of the octagon consists of d digits.

S.No.	n	Number of digits in n!
1	33	37
2	85	129
3	245	481
4	350	741
5	471	1057
6	681	1636
7	924	2341
8	1012	2604
9	1297	3477
10	1399	3796
11	1613	4476
12	1725	4837
13	1959	5601
14	2081	6004
15	2206	6421
16	2601	7756
17	2739	8229
18	4464	14356
19	4639	14997

Table 4

868
33176
1881188
6495518
1944012
80000
000

33!

28171
0411438
055027694
94794422606
1159480056634
3305742064051
0191275256002
6159795933451
0402864523409
24018275123
200000000
0000000
00000

85!

344638108
85481846673
2677079087595
192200697695137
55823846110036119
8163952978235186876
380414323767237937984
68686285775275996090439
4401084934335518382691657
9274876093562728501050027
8210756098320353036549967
4936175319516301276907070
0485723141551669720900953
7280115585840030353759282
1265584694028136706112564
5619508966404523085638743
7632569808704944651568598
19224514274661087972929
242817912092409671474
4916317976775473389
10924800000000000
000000000000000
0000000000000
00000000000
000000000

245!

Figure 4

Isosceles Triangle from factorial digits

For an isosceles triangle, it may be seen that the first row consists of 1 digit, 2nd row consists of 3 digits, 3rd row consists of 5 digits and so on. So, n^{th} row consists of $(2n - 1)$ digits. So, the number of digits is the partial sum of series 1 + 3 + 5 + 7 + 9 +…. which is always a perfect square. Can you use the factorials in Table 2 to draw isosceles triangles?

Using factorial digits, it is possible to draw various other shapes such as cross (+), E, F, I, L, T etc. It will be an interesting pastime to attempt to draw the above mentioned shapes and find out the factorials with the required number of digits for the desired shapes.

SHYAM SUNDAR GUPTA served in the Indian Railways for 35 years, and retired as Principal Chief Engineer in 2018. Popularising mathematics through number recreations has been his passion for more than forty years. His contributions have been published in National and International journals/books. He is the author of the book "Creative Puzzles to Ignite Your Mind" published by Springer in March 2023 and co-author of the book 'Civil Engineering through Objective Type Questions' published in 1985. E-mail: guptass@rediffmail.com Web page: http://www.shyamsundergupta.com

How Much or Till What: When and Why?

MATH SPACE

One of the decisions policy makers, syllabus and textbook writers need to consider for mathematics at the primary level (Class 1-5) or foundational and preparatory stages (pre-school to Class 5 or 3-11yrs), is what is the biggest number children should be introduced to at various classes. The National Curricular Framework for Foundational Stage (NCF-FS) has made its recommendations. Here is another take at that along with some justification and interlinkages with other topics within the subject.

Numbers have essentially 3 aspects – (i) the quantity each represents, usually the cardinality of a set for whole numbers, (ii) the number name (e.g., thirteen or hundred and five) and (iii) the numeral (e.g., 13 or 105). Learners are supposed to establish a 3-way mapping linking these three aspects together. It may be helpful to separate out (i) and (ii) before connecting (iii) with the same, at least initially.

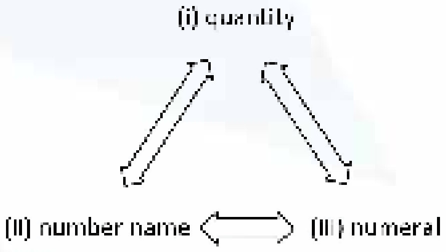


Figure 1

Keywords: Number sense, numerals, number names, math pedagogy

Usually, the sequence is as follows:

Class 1	• 1-5 or 1-9 and possibly some addition-subtraction • 0 • 0-20 along with addition-subtraction with these numbers • Exposure to 0-100
Class 2	Extensive engagement with 0-100 – place value and addition-subtraction
Class 3	Playing with 0-999 or 0-1000
Class 4	0-9999
Class 5	0-99999
Class 6+	even bigger numbers

The usual tussles include whether to stop at 9, 99, 999 etc. or to go till 10, 100, 1000 etc. Here are our suggestions and the justifications for the same:

One to Ten: This is connecting (i) the quantities with (ii) the number names. Numerals don't come into this part. Children are essentially learning to count. We would say that a child can count till ten if,

1. S/he remembers the number name sequence – one, two… ten
2. S/he correctly counts a given collection of ten or fewer objects
3. Given a number ≤ ten, s/he picks up exactly that many objects from a larger collection

Fingers are very useful in counting. Most of us have 10 fingers and that is considered to be the reason that we have a base-10 system (place value) of writing numbers. So, it feels artificial to stop at nine.

The main reason for excluding ten is that it is a 2-digit number. But if we consider connecting quantity with number name (and leave out the numeral aspect) then that reason is no longer relevant.

1-9: Based on the knowledge children have acquired, they can be nudged to extend their understanding of numbers to the numerals for 1-9. This can be achieved using double sided cards, one side showing the numeral while the other show the quantity as dots.

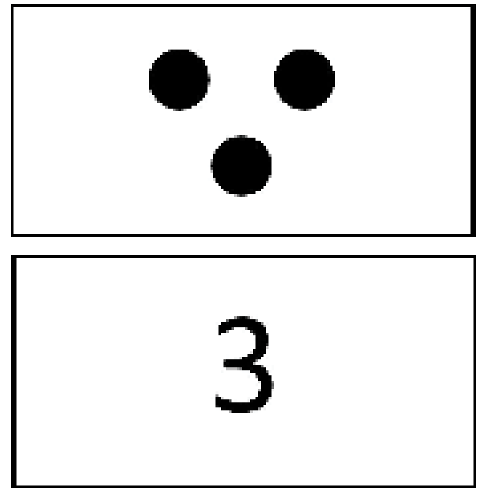


Figure 2

It is advisable to initiate addition-subtraction after this as is done in several textbooks.

Zero (0): Introduction or zero as nothing and the numeral 0 representing the same. This can be done as showing a quantity decrease by one gradually till nothing is left (as done in several textbooks). And then the numeral can be introduced using a similar card with 0 on one side and nothing, i.e., no dot on the other.

0-20: This should be done in several steps as follows:

a. Connecting quantity with number names for eleven to fifteen (say)

b. Initiate bundling, first in twos, threes, etc., and then in tens

c. Discuss numeral of 11-15 as number of bundles followed by number of loose sticks outside the bundle, ganitmala can be used to discuss why the number of tens, i.e., the ten's digit is on the left of the number of ones, i.e., the one's digit (see reference for further details)

d. Discuss what the numeral for ten should be by problematizing and drawing upon children's knowledge of zero

e. Extend understanding of numbers 0-20

It is advisable to continue addition-subtraction with 0-20 after this. It is possible to introduce doubling and halving (as divide equally in two groups) at this point as well.

0-100: Extend numbers further with bundles and loose sticks. It makes sense to stop at 10 tens, i.e., hundred and give children a glimpse at how bigger bundles must come in when we have 10 of a kind. Hundred is a number that is encountered quite often in day-to-day lives as currency and as century (in a cricket loving country in particular). Including 100 provides an opportunity to understand how the numerals are obtained for bigger and bigger numbers – one of the basic ideas of place value: if there are ten of a kind (loose sticks or bundles), make a new (and bigger) bundle.

The chart of 1-100 with ten rows and ten columns is also a widely used teaching aid. Stopping at 99 would create a hole in that.

0-1000: Thousand is the next bigger bundle, i.e., 10 hundreds. And is used in the standard units with respect to kilo and milli (1km = 1000m, 1kg = 1000g, 1m = 1000mm, 1g = 1000mg and 1l = 1000ml). Therefore, thousand is also a number associated deeply with day-to-day life. So, instead of stopping at 999, it makes sense to go till 1000 especially if standard units are discussed in the same class.

Inclusion of 1000 also allows children to observe the above-mentioned basic idea of place value play out with another set of bundles, viz., hundreds.

This is where subtraction poses newer difficulties for problems like 500 – 162. Also, where all four operations get introduced.

Beyond thousands, the next bundle name is lakh. It comes after bundling thousands twice;

(10 thousands = 1 ten-thousand, 10 ten-thousand = 1 lakh) and is not used that often in day-to-day lives. So, we feel it is ok to restrict to 4-digit (i.e., up to 9999) and 5-digit (till 99,999) numbers. This is also because unless lakh is introduced, the main implication of these restriction is limited to the numbers used in the four operations, specifically, sums, products and dividends should not exceed 9999 or be within 4-digit. Same restriction applies to minuend or the first (or bigger) number in a subtraction. The need for sums, minuends, products or dividends beyond 4-digits does not occur that naturally in the contexts of primary mathematics.

One should also keep in mind that these restrictions (till 999 or till 1000) are possibly needed at the policy level to provide guidance to curriculum/syllabus/textbook developers, teachers and to track progress of children's learning levels in a large system. These do NOT come from mathematics. In fact, mathematically speaking, numbers naturally grow especially as children start playing with them, and explore further (e.g., palindromic numbers). So, if children are able to engage with multi-digit numbers with ease, i.e., add-subtract and possibly also multiply-divide, then a teacher should not put an artificial restriction on the largest number they encounter.

Acknowledgements

We would like to acknowledge Rohit Dhankar, founder-secretary of Digantar, for some of the key ideas especially on how to introduce the numeral of ten through problematization.

We are grateful to Padmapriya Shirali for valuable input on this topic.

References:

Swati Sircar: 'Why TLM – its aims and uses', *Learning Curve*, Issue 3, December 2018, p.56

MATH SPACE is a mathematics laboratory at Azim Premji University that caters to schools, teachers, parents, children, NGOs working in school education and teacher educators. It explores various teaching-learning materials for mathematics [mat(h)erials] – their scope as well as the possibility of low-cost versions that can be made from waste. It tries to address people at both ends of the spectrum, those who fear or even hate mathematics as well as those who love engaging with it. It is a space where ideas generate and evolve thanks to interactions with many people. Math Space can be reached at mathspace@apu.edu.in

Math Space: https://sites.google.com/apu.edu.in/mathspace/home

Sums of Powers of Any Composite Number

SUJATA SINGHA

In 1997, David B. Sher visually established a well-known result on sums of powers of four [1, p.135]. He proved for any $r \in \mathbb{N}$,

$$4^r - 1 = 3\sum_{k=0}^{r-1} 4^k$$

i.e.,

$$\frac{4^r - 1}{4 - 1} = \sum_{k=0}^{r-1} 4^k$$

In 2021, extending the same technique in three dimensions another results for sums of powers of eight [2] is established.

$$8^r - 1 = 7\sum_{k=0}^{r-1} 8^k$$

i.e.,

$$\frac{8^r - 1}{8 - 1} = \sum_{k=0}^{r-1} 8^k$$

The primary aim of this short note is to visually establish sums of powers of any composite number.

For $9 = 3 \times 3$, we illustrate the above result in the following diagram. The above diagram also generalizes the result David B. Sher.

Keywords: Series, Composite numbers, patterns, visualisation.

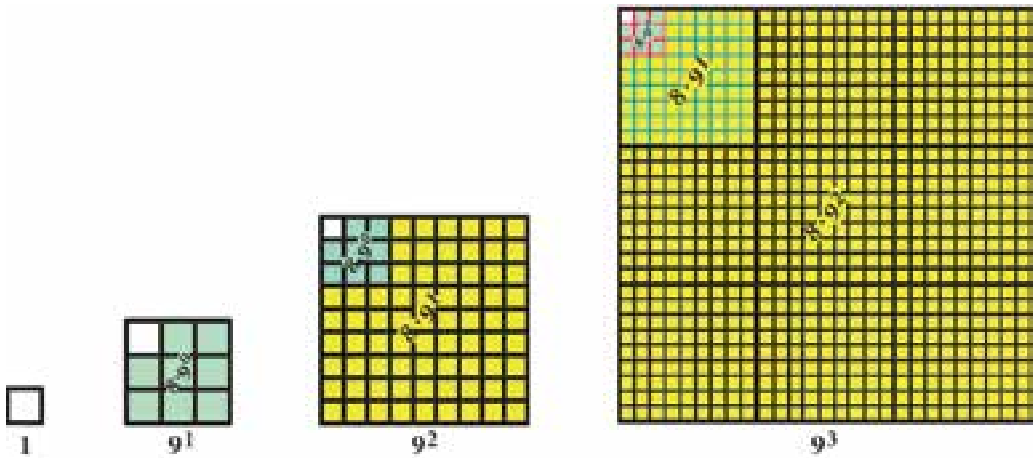


Figure 1

Note that, in the 9^3 grid, 8×9^2 is represented by 8 squares with $9^2 = 81$ smaller squares.

$$9^r = 1 + 8.9^0 + 8.9^1 + 8.9^2 + \ldots + 8.9^{r-1}$$

$$\Rightarrow 9^r - 1 = 8\left[9^0 + 9^1 + 9^2 + \ldots + 9^{r-1}\right]$$

$$\Rightarrow \frac{9^r - 1}{8} = \sum_{i=0}^{r-1} 9^i,$$

$$\sum_{i=0}^{r-1} 9^i = \frac{9^r - 1}{9 - 1}$$

Note: Let $x = m \times n$, be any composite number, m, n be two natural numbers, then

$$\sum_{i=0}^{r-1} x^i = \frac{x^r - 1}{x - 1}$$

Now, I extend the above result in 3D for the composite number 27.

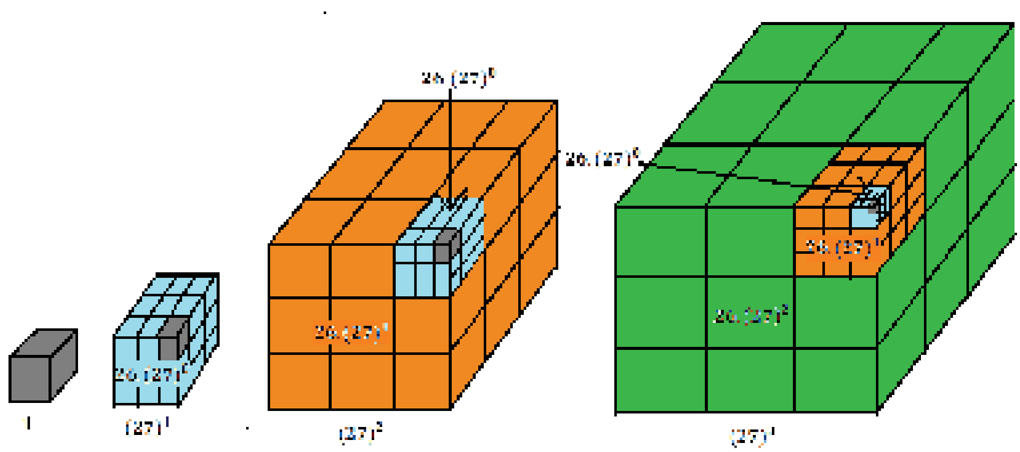


Figure 2

For $27 = 3 \times 3 \times 3$, we illustrate the above result in Figure 2.

Please note that, in the 27^3 grid, 26×27^2 is represented by 26 cubes with $27^2 = 729$ smaller cubes.

$$(27)^r = 1 + 26.\,(27)^0 + 26.\,(27)^1 + 26.\,(27)^2 + \ldots + 26.\,(27)^{r-1}$$
$$\Rightarrow (27)^r - 1 = 26\left[(27)^0 + (27)^1 + (27)^2 + \ldots + (27)^{r-1}\right]$$
$$\Rightarrow \frac{(27)^r - 1}{26} = \sum_{i=0}^{r-1} (27)^i$$

Note: Let $x = m \times n \times p$, be any composite number, m, n, p be any three natural numbers. Then,

$$\sum_{i=0}^{r-1} x^i = \frac{x^r - 1}{x - 1}$$

For any composite number $x = m_1.m_2 \ldots m_k$, one can use similar techniques to generalize the above two results in k-dimensions $(k \geq 2)$.

In this short note, I have tried to generalize [1] and [2] for any composite x with $x = m_1.m_2 \ldots m_k$.

$$\sum_{i=0}^{r-1} x^i = \frac{x^r - 1}{x - 1}$$

Reference

1. Roger B. Nelsen. Proofs without Words III: More Exercises in Visual Thinking. MAA Press, 2015.
2. Rajib Mukherjee, Proofs without Words: Sums of Powers of Eight Mathematical Intelligencer **43**, 60-61 (2021) (https://doi.org/10.1007/s00283-021-10087-5).

SUJATA SINGHA is currently pursuing a Master of Science in Mathematics at Murshidabad University. She graduated from Kalayani University's Berhampore Girls' College with a B.Sc. in Mathematics. She is interested in visual mathematics. Outside of studies, she spends time with friends and family. She may be contacted at sujatasingha25@gmail.com.

Nesting Platonic Solids

VANSHIKA MITTAL

What are Platonic Solids?

Platonic solids are a type of regular convex polyhedra, that are made up of a number of regular faces meeting at a vertex. There are five known platonic solids, namely the dodecahedron, cube, tetrahedron, octahedron, and the icosahedron.

What are nested platonic solids?

The idea of nested platonic solids involves perfectly placing or 'fitting' platonic solids within each other. There are multiple combinations to nest the five platonic solids. One of these ways include nesting them in the following order:

- an icosahedron sits inside an octahedron with each of the 12 vertices of the icosahedron touching an edge of the octahedron,
- the octahedron is placed inside the tetrahedron, with each alternate face touching the equilateral triangle in the centre of each face of the tetrahedron,
- the tetrahedron is placed in a cube with four of its vertices touching diagonal vertices of two parallel square bases,
- which, i.e., the cube, finally nests in a dodecahedron with two horizontal diagonal vertices of every pentagon coinciding with the vertices of the cube.

Keywords: Platonic Solids, nesting solids, geometry, congruence, golden ratio.

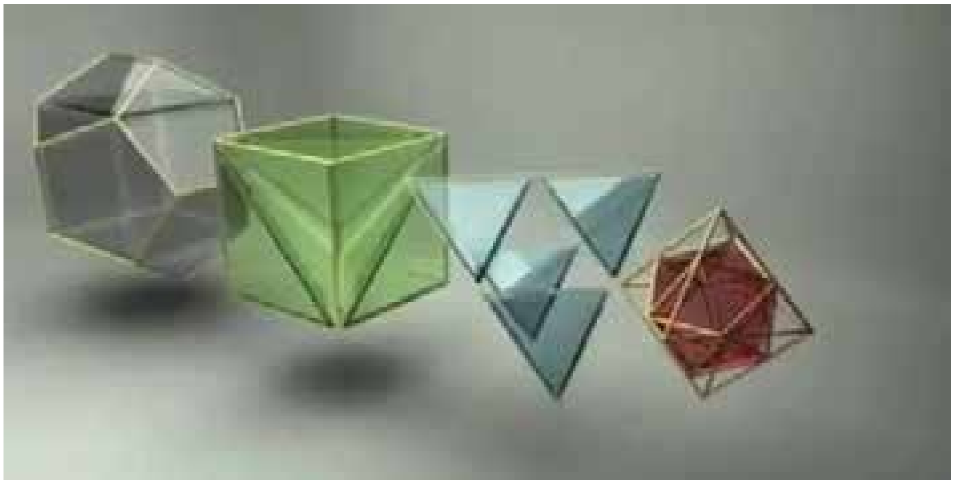


Figure 1. (Source: https://youtu.be/gwxQfujwWrw)

So, what dimensions work to nest the platonic solids in order?

To begin with nesting platonic solids, one needs to decide the dimension of the innermost solid. Intuitively, bigger the dimension of the innermost solid, larger will be the subsequent solids as the ratio between the dimensions of the solids remains fixed in order to nest them in a given way. I started out by constructing an icosahedron of side 4cm. To find dimensions of subsequent nesting solids, I watched this video (https://youtu.be/gwxQfujwWrw) multiple times to fully understand how the icosahedron sits inside the next solid, i.e., the octahedron.

During initial attempts, despite rewatching the video multiple times, some of my doubts about the placement of the icosahedron inside the octahedron remained. So, I decided to make a dummy octahedron of double the dimensions of that of the icosahedron[1]. This helped me understand how the two solids existed, one within the other. It also provided perspective on what the dimensions of the octahedron could be, as I was able to trace the boundary of the octahedron where the top half was supposed to close in order for the icosahedron to sit inside completely without any extra space or gaps.

After observing, understanding and confirming the fact that an icosahedron nested by an octahedron has each of its vertices touching a unique edge of the octahedron, I proceeded to work on closer approximations, considering 2D and 3D orientations.

To find the side of the octahedron, consider the orientation of the icosahedron in the top view of the nesting. The vertices of the icosahedron touch the edge of the octahedron in two ways.

(1) In the first possible way, the two vertices, e.g., G and H, of the icosahedron touch two adjacent edges of the octahedron, in which case the edge, e.g., GH, of the icosahedron spans the distance s between the two contact points on the boundary ('perimeter' of the square base $CD_1E_1F_1$) of the octahedron.

1 Intuitively, the octahedron of double the dimensions will be able to accommodate the chosen icosahedron. Consider the following: if a section of the icosahedron were to lay flat on a plane, then the height of the section in that case will be given by 3 times the median length of the equilateral triangle (building unit of the icosahedron), i.e., $3\left(\frac{\sqrt{3}}{2}\right)4 = 6\left(\sqrt{3}\right)$. This approximation will also give a larger dimension for the enclosing octahedron because it is derived from the assumption that the section lies in the plane. However, it tells us that an octahedron of dimension 8 cm (i.e., twice the side of the icosahedron and greater than the obtained length on the plane) will also be able to fully contain the icosahedron.

(2) In the second way, the vertex (e.g., L) of the icosahedron sits on one of the inner edges (either above or below the base square) of the octahedron.

In the former case, a right-angled isosceles triangle ($\triangle CGH$, right angled at C) is obtained at the corner, as the icosahedron is uniformly seated inside the octahedron.

The latter case (e.g., L) also forms an isosceles right-angled triangle with legs h (e.g., D_1K and D_1G) and hypotenuse d (diagonal of the 2D pentagon, e.g., $LGMNK$, formed on top of the icosahedron), e.g., diagonal GK forms $\triangle GD_1K$, right angled at D_1. The sum of two different legs, i.e., $s + h = D_1G + GC$, from these two right isosceles triangles gives the side of the octahedron D_1C.

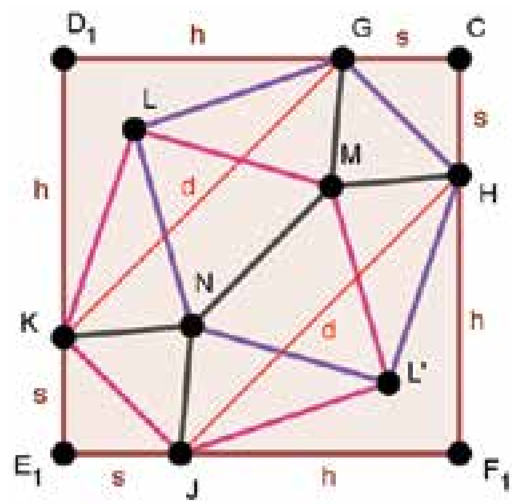


Figure 2. Top view of nesting an icosahedron inside an octahedron - $CD_1E_1F_1$ is a square base of the double pyramid, i.e., octahedron.

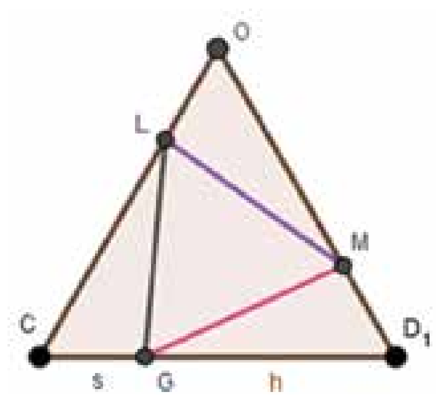


Figure 3. Front view of an icosahedron placed inside an octahedron - icosahedron face overlapping an octahedron face.

When a face of the octahedron with the embedded face of the icosahedron in Figure 3 is considered, the congruence of triangles $\triangle GMD_1$ (PQB), $\triangle LGC$ (RPA) and $\triangle MLO$ (QRO) can be looked at using rotational symmetry. Consider without loss of generality, $\triangle GMD_1$ and $\triangle LGC$. It is known that $GM = GL$ (PQ = PR) (sides of an equilateral triangle). By rotating the equilateral triangles $\triangle CD_1O$ (ABO) and $\triangle GML$ (PQR) 120° clockwise about their common centre it can be concluded without ambiguity that $GD_1 = CL$ (PB = AR) $= h$ and $MD_1 = GC$ (QB = PA) $= s$ as the two triangles superimpose on each other (Figure 3). Therefore, $\triangle GMD_1 \cong \triangle LGC \cong \triangle MLO$, by SSS criterion using rotational symmetry of equilateral triangles.

In Figure 2, the pink and purple line segments show two pentagonal sections of an icosahedron, depicted three-dimensionally.

Length of side of octahedron $= s + h$ where $s = CG$ and $h = GD_1$ from Figure 2.

From the right isosceles $\triangle CGH$,

$$2s^2 = 16 \Rightarrow s^2 = 8 \Rightarrow s = 2\sqrt{2}$$

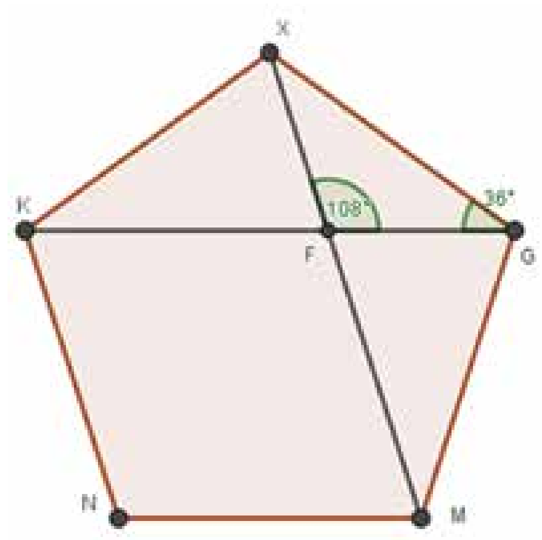


Figure 4

In Figure 4 above, the four points of the pentagon K, N, M, G are obtained using Figure 2, while X is the fifth vertex lying behind and joining L, which is not visible in the two-dimensional Figure 2. So, side of the icosahedron $= KN = KX = GX = 4$.

In Figure 4, the acute angles of $\triangle GFX$ and $\triangle GXK$ are equal (angles adjacent to bases of $\triangle GXK \cong \triangle MGX$). Therefore,

(1) The remaining angles are equal, i.e., $\measuredangle GFX = \measuredangle GXK = 108°$ (internal angle of a regular pentagon)

$\Rightarrow$The acute angles, viz. $\measuredangle XGK = \measuredangle XKG = \measuredangle FXG = \measuredangle FGX = \frac{1}{2} \times \measuredangle XFG = \frac{1}{2}(180° - \measuredangle XFG)$

$= \frac{1}{2}(180° - 108°) = 36°$

$\Rightarrow \measuredangle KXF = 180° - (\measuredangle XFK + \measuredangle XKF) = 180° - (72° + 36°) = 72° = \measuredangle XFK$

$\Rightarrow KF = KX$ (opposite sides of equal angles are equal) $= 4$

(2) $\triangle GFX \sim \triangle GXK$

$\Rightarrow$ by proportional side argument of similar triangles $\frac{FG}{GX} = \frac{GX}{GK}$.

Recall that GK $= d$ from Figure 2. Therefore, $FG = GK - KF = d - 4$. So, $\frac{d-4}{4} = \frac{4}{d}$

$\Rightarrow d(d-4) = 16 \quad \Rightarrow d^2 - 4d - 16 = 0$

$\Rightarrow d = \frac{1}{2}\left(4 \pm \sqrt{16 - 4(-16)}\right) = 2 \pm 2\sqrt{5} = 2\left(1 + \sqrt{5}\right)$

Therefore, in the right isosceles $\Delta\ GD_1K$, with leg $D_1G = h$ and diagonal $GK = d$, $2h^2 = d^2$

$\Rightarrow h = \frac{d}{\sqrt{2}} = \sqrt{2}\left(1+\sqrt{5}\right)$

So, side of octahedron is $s + h = 2\sqrt{2} + \sqrt{2}\left(1+\sqrt{5}\right) = \sqrt{2}\left(3+\sqrt{5}\right)$.

Based on the nesting of the octahedron inside the tetrahedron, the side length of tetrahedron spans twice that of the octahedron, which gives side of the tetrahedron

$$= 2(s+h) = 2\left(\sqrt{2}\left(3+\sqrt{5}\right)\right).$$

Figure 5

The side of the tetrahedron spans the diagonal of the enclosing cube. So, if side of cube is s_c, then $\sqrt{2}s_c = 2(s+h) \Rightarrow s_c = \sqrt{2}\left(s+h\right) = 2\left(3+\sqrt{5}\right)$.

Figure 6

The side of the cube s_c is equal to the diagonal of the pentagonal face of the dodecahedron, which means that the side of the dodecahedron s_d is given using the relation between the side and diagonal of the pentagon.

$\Rightarrow \frac{s_d}{s_c} = \frac{2}{\sqrt{5}+1}$ (from golden ratio in a pentagon)

Therefore, side of dodecahedron

$$= \frac{2}{\sqrt{5}+1} s_c = \frac{2}{\sqrt{5}+1}\sqrt{2}\,(s+h) = \frac{\sqrt{2}\left(\sqrt{5}-1\right)}{2}\,(s+h) = \frac{1}{2}\sqrt{2}\left(\sqrt{5}-1\right)\left(\sqrt{2}\left(3+\sqrt{5}\right)\right)$$
$$= 2\left(\sqrt{5}+1\right)$$

Figure 7

This entire calibration is dependent on understanding the orientation of the icosahedron inside the octahedron. Once that is understood by the reader, the remaining calculations involve simple geometry arguments.

A very interesting conclusion presents itself as one proceeds to find the dimensions of all subsequent platonic solids. It is found that these dimensions exist obeying the famous golden ratio! This stems from the fact that the golden ratio is observed in a regular pentagon, as the ratio between the side and diagonal of the pentagon.

A set of Possible Nets

1. Octahedron

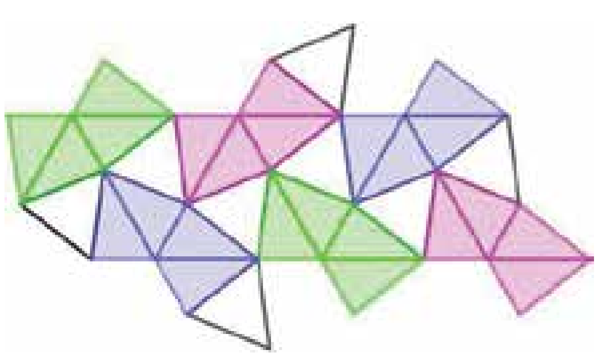


Figure 8

2. Tetrahedron

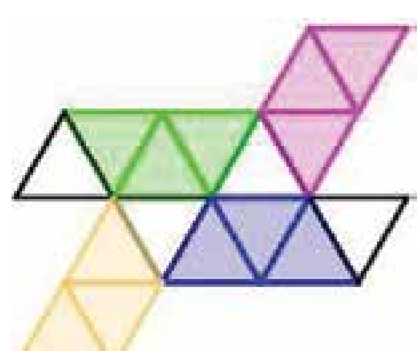


Figure 9

3. Cube

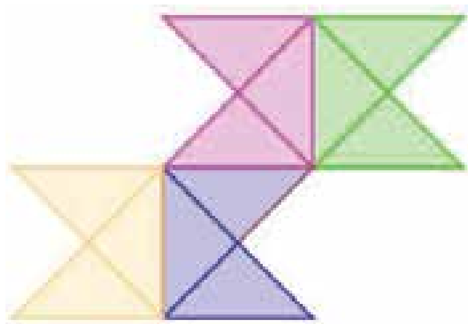


Figure 10

4. Dodecahedron

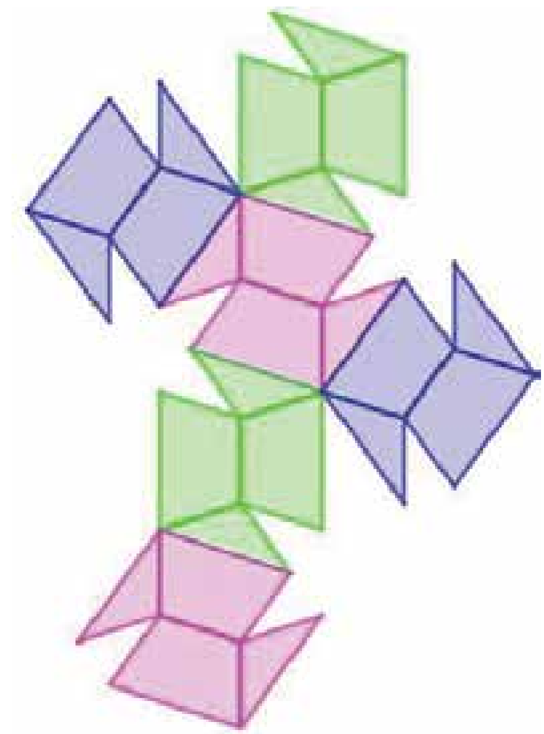


Figure 11

To create the above set of nets, I took the help of the referenced video to understand how each of the solids was seated within the other. It was an interesting task to trace possible paths to arrive at these nets while having known only how they were nested within one another. After finding a possible net for the octahedron, for example, a pattern of alternating triangles used to generate the entire solid was observable. Such a pattern could be replicated for other nets as well (except the dodecahedron where trapeziums also had to be included to form the pentagons).

References

1. https://youtu.be/gwxQfujwWrw
2. http://nonagon.org/ExLibris/sites/default/files/pdf/Kepler-Nested-Platonic-Solids.pdf

Note from Math Space

The author, Vanshika, explored the nested Platonic solids based on the video she mentioned above. The assignment involved understanding the nesting for each pair of solids and thereby finding the relative lengths of the sides of each pair. But she went beyond that! She came up with two nets, viz. those of the tetrahedron and the cube such that the coloured triangles of each net reveal the nesting of the corresponding pair of solids. This inspired us to take the work forward and create the remaining two nets, those of the octahedron and the dodecahedron. In doing the above we observed how the usual nets can be partitioned and then the resulting polygons moved around to cluster together colour-wise. Here are the complete set of nets – the usual and the new:

Octahedron net

Sides of equilateral triangles = medium sides of acute scalene triangles = 4cm

Shortest sides of scalene triangles $= 2\sqrt{2}$cm ≈ 2.83cm

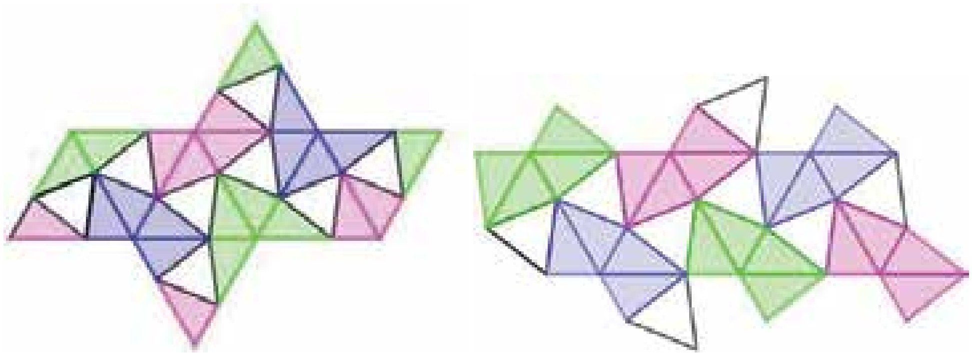


Figure 12

Tetrahedron net

Sides of smallest equilateral triangles $= \sqrt{2}\,(3+\sqrt{5})$ cm ≈ 7.40cm

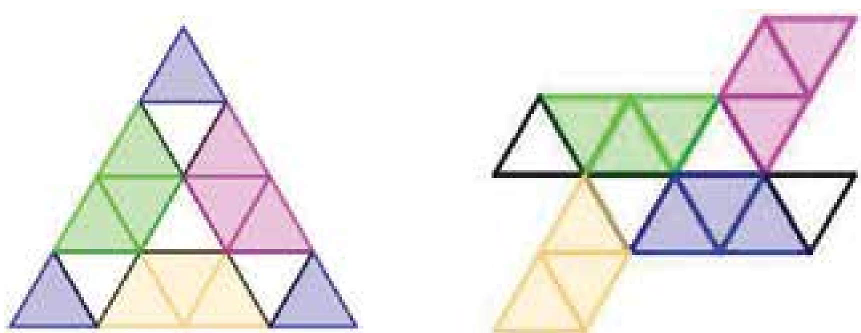


Figure 13

Cube net

Sides of squares = legs of right isosceles triangles $= 2\,(3+\sqrt{5})$ cm ≈ 10.47cm

Diagonals of squares = hypotenuse of right isosceles triangles $= 2\sqrt{2}\,(3+\sqrt{5})$ cm ≈ 14.81cm

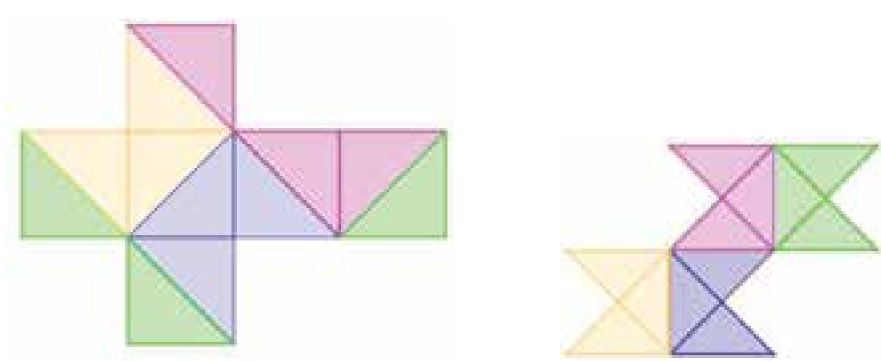


Figure 14

Dodecahedron net

Sides of regular pentagons = shorter sides of obtuse isosceles triangles = shorter sides of isosceles trapeziums $= 2\left(3+\sqrt{5}\right)$ cm $\approx$ 10.47cm

Diagonal of regular pentagons = longest side of obtuse isosceles triangles = longest side of isosceles trapeziums $= 2\left(1+\sqrt{5}\right)$ cm $\approx$ 6.47cm

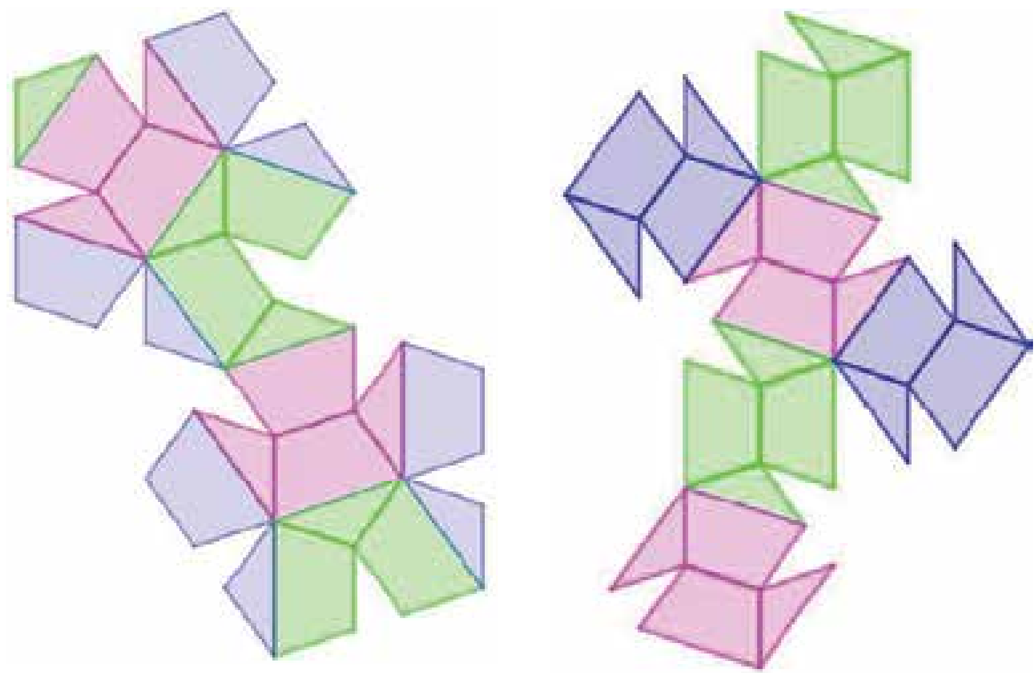


Figure 15

These nets enabled us to make physical models of the solids in question. The earlier attempt to make such models involved making the following solids separately.

(1) Icosahedron

(2) Octahedron: 6 concave hexahedrons with 2 consecutive faces as equilateral triangle, identical to the faces of the icosahedron, and the remaining 4 faces as congruent acute scalene triangles

(3) Tetrahedron: 4 regular tetrahedrons each with sides equal to that of the octahedron

(4) Cube: 4 pyramids with equilateral triangle, whose sides equal to those of the tetrahedron, as base and right isosceles triangles as the remaining faces

(5) Dodecahedron: 6 pentahedrons with square bases, identical to the faces of the cube, the remaining being 2 pairs of congruent polygons – isosceles trapeziums and obtuse isosceles triangles, alternating with each other – all remaining sides of these 4 polygons being equal to the sides of the dodecahedron.

Now, these pentahedrons can be taped along some edges to form a shell that encloses the cube and generates the dodecahedron. Similarly, the pyramids can be joined to form a shell enclosing the tetrahedron and making the cube. The 4 smaller tetrahedrons can be placed around and above the octahedron to generate the tetrahedron inside the cube. But it is quite difficult to connect the hexahedrons to form the shell for the octahedron. This is mainly because these concave solids connect only at some of their vertices. This problem is resolved thanks to the nets improvised and inspired by the author. Note that the same acute scalene triangles are in the nets in Figure 12 as well. The rest of the polygons (isosceles and equilateral triangles and isosceles trapeziums) are all symmetric and therefore easier to construct. But the construction of this acute scalene triangles revealed some unexpected surprises! As illustrated in Figure 3, the shortest side, e.g., AP of this triangle is $2\sqrt{2}$cm while the medium side is 4cm. Also, the angle opposite to the medium side is $60°$. Now, $2\sqrt{2}$cm can be easily obtained from 4cm using a right isosceles triangle. So, by Side-Side-Angle (SSA) construction, one can easily generate this acute scalene triangle. On the other hand, the longest side is $\sqrt{2}\left(1+\sqrt{5}\right)$ which is more complicated to construct. So, SSA is a preferred way than SSS. Note that SSA construction (where (i) an angle, (ii) an adjacent side and (iii) the opposite side are given) is no longer part of many textbooks. This is possibly because it fails to be a congruency criterion like the rest, i.e., SAS, SSS, ASA (and AAS) and RHS. If the given angle is acute and the side opposite to it is shorter than the given adjacent side, then there can be 2 triangles – one acute, e.g., $\triangle ABC$ and one obtuse, e.g., $\triangle ABD$ both satisfying the SSA criterion $\measuredangle A, AB, BC = BD$ (see Figure 16). In this case however, since the adjacent side $2\sqrt{2}$cm < 4cm, the opposite side, the construction generates a unique triangle. One can easily find applications of SAS, SSS, ASA (AAS) and RHS in constructing various quadrilaterals, in particular. We were pleasantly surprised to find an application of SSA while exploring nested Platonic solids!

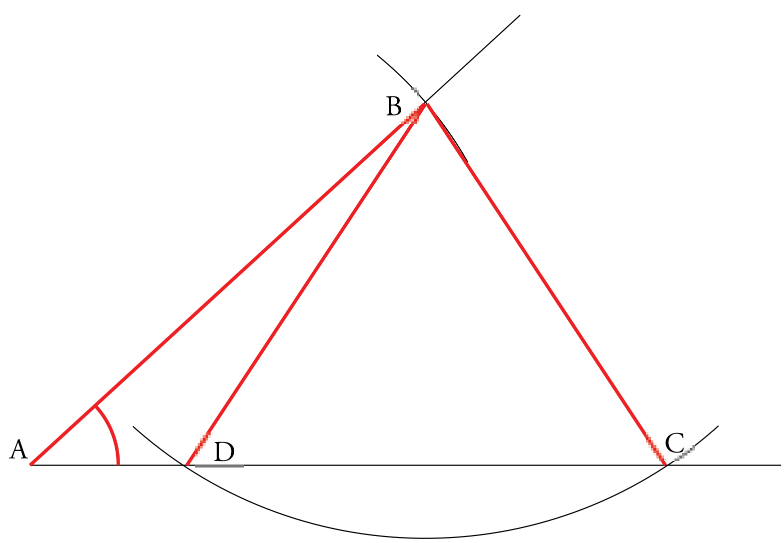


Figure 16

VANSHIKA MITTAL is currently a third year undergraduate student pursuing B.Sc (Honours) Mathematics at Azim Premji University, Bangalore, though she worked on this project at the end of her first year. She is currently working on a project to understand the concept of Fractional Calculus as part of the Honours program at the university. She finds topics in Analysis particularly interesting to study. Apart from mathematics, she also likes playing the violin and understanding patterns between Mathematics and music. Vanshika may be contacted at vanshikamittal64@gmail.com

On a Link between Three Trigonometric Identities for a Triangle

DR. JYOTI NEMA &
DR. POONAM AGGARWAL

For a ΔABC we have the following three important and powerful trigonometric identities:

I_1. $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

I_2. $c = a\cos B + b\cos A$, and similar identities for a and b.

I_3. $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$, and similar identities for $\cos A$ and $\cos B$.

In this short note we show that any one of these results can be used to prove the other two (using only standard trigonometric manipulations and without the need for any diagram).

Proof:

Let $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = \dfrac{1}{k}$, where k represents a non-zero constant.

Keywords: sine rule, cosine rule, proofs, connections

To prove $I_1 \implies I_2$:

We have,

$$\begin{aligned} c &= k \sin C \\ &= k \sin(A+B) \\ &= k[\sin A. \cos B + \cos A. \sin B] \\ &= (k \sin A). \cos B + (k \sin B) \cos A \\ &= a \cos B + b \cos A. \end{aligned}$$

To prove $I_1 \implies I_3$:

From $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{k}$ we get:

$$\begin{aligned} & a^2 + b^2 - c^2 \\ &= k^2 \sin^2 A + k^2 \sin^2 B - k^2 \sin^2 C \\ &= k^2 \left(\sin^2 A + \sin^2 B - \sin^2 C\right) \\ &= k^2 \left[\sin^2 A + \sin(B+C) \sin(B-C)\right] \\ & \quad \text{(we justify this step in the Appendix)} \\ &= k^2[\sin A \sin(B+C) + \sin A \sin(B-C)] \\ &= k^2 \sin A[\sin(B+C) + \sin(B-C)] \\ &= k^2 \sin A \cdot 2 \sin B \cdot \cos C \\ &= 2ab \cos C, \end{aligned}$$

therefore

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

To prove $I_2 \implies I_3$

We have,

$$\begin{aligned} & a^2 + b^2 - c^2 \\ &= a.a + b.b - c.c \\ &= a(b \cos C + c \cos B) + b(a \cos C + c \cos A) \\ & \quad - c(a \cos B + b \cos A) \\ &= 2ab \cos C \\ & \Rightarrow \cos C = \frac{a^2 + b^2 - c^2}{2ab} \end{aligned}$$

To prove $I_2 \implies I_1$

We have,

$$\begin{aligned} c &= a \cos B + b \cos A, \\ a &= b \cos C + c \cos B. \end{aligned}$$

Therefore:

$$\begin{aligned} & c^2 - a^2 \\ &= c \cdot c - a \cdot a \\ &= c(a \cos B + b \cos A) - a(b \cos C + c \cos B) \\ &= b(c \cos A - a \cos C) \\ &= (c \cos A + a \cos C) \cdot (c \cos A - a \cos C) \\ &= c^2 \cos^2 A - a^2 \cos^2 C. \end{aligned}$$

Hence:

$$\begin{aligned} c^2\left(1 - \cos^2 A\right) &= a^2\left(1 - \cos^2 C\right), \\ \therefore c^2 \sin^2 A &= a^2 \sin^2 C, \\ \therefore c \sin A &= a \sin C. \end{aligned}$$

(Since A, B, C are the angles of a triangle, $\sin A, \sin B, \sin C$ are positive quantities; no negative signs are introduced when we take square roots.) From the last we get

$$\frac{\sin A}{a} = \frac{\sin C}{c}.$$

By symmetry it follows that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

To prove $I_3 \implies I_1$

We have,

$$\begin{aligned} & (2bc \sin A)^2 \\ &= 4b^2c^2\left(1 - \cos^2 A\right) \\ &= 4b^2c^2\left(1 - \left(\frac{b^2 + c^2 - a^2}{2bc}\right)^2\right) \\ &= 4b^2c^2 - (b^2 + c^2 - a^2)^2 \\ &= (2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2) \\ &= \left\{(b+c)^2 - a^2\right\}\{a^2 - (b-c)^2\} \\ &= (a+b+c)(b+c-a)(c+a-b)(a+b-c). \end{aligned}$$

Then is symmetrical in a, b, c

Hence by symmetry
$2bc \sin A = 2ac \sin B = 2ab \sin C$, so

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

To prove $I_3 \Longrightarrow I_2$

We have,

$$\begin{aligned} & a\cos B + b\cos A \\ &= a\cdot\frac{c^2+a^2-b^2}{2ac} + b\cdot\frac{b^2+c^2-a^2}{2bc} \\ &= \frac{1}{2c}\left(c^2+a^2-b^2+b^2+c^2-a^2\right) \\ &= c. \end{aligned}$$

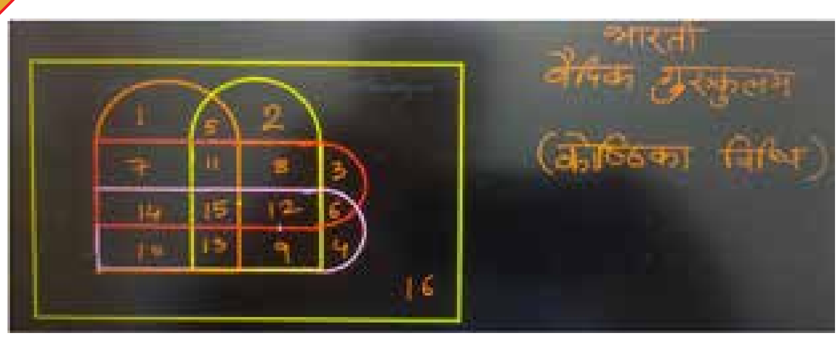
Appendix

$$\begin{aligned} & \sin^2 B - \sin^2 C \\ &= (\sin B + \sin C)(\sin B - \sin C) \\ &= 2\,\sin\frac{B+C}{2}\cos\frac{B-C}{2}\cdot 2\,\sin\frac{B-C}{2}\cos\frac{B+C}{2} \\ &= 2\,\sin\frac{B+C}{2}\cos\frac{B+C}{2}\cdot 2\,\sin\frac{B-C}{2}\cos\frac{B-C}{2} \\ &= \sin\left(B+C\right)\sin\left(B-C\right). \end{aligned}$$

DR. JYOTI NEMA (MCA, Ph.D Mathematics, MPSLET) is Assistant Professor Mathematics in Regional Institute of Education NCERT Bhopal (Ex Faculty). She has 20 years of teaching experience in the field of Mathematics. She has published 13 research papers (both national and international) and served as resource person in many initiatives in the pedagogy of mathematics. She has also worked with Mathematics Software such as MATLAB, Mathematica, Geogebra, Robocompass, LOGO,C++, etc.

DR. POONAM AGGARWAL (M.Sc (Mathematics), M.A (Economics and Education), M.Ed., NET-JRF in Education and Ph.D in Education) works as Assistant Professor, Regional Institute of Education NCERT Bhopal. She has 12 years of teaching experience. She has published 18 research papers in various national and international journals and served as resource person in many activities of Mathematics Education. She provided guidance and support services to adolescents at PM e-Vidya Platform.

VENN DIAGRAM



What is the universal set?

How have the elements of the universal set been arranged in the four sets shown?

ClassRoom

Radius (त्रिज्या) and Sine (ज्या) – a study of the Names and their Relationship

DR. KOMAL ASRANI

India has had a long-standing relationship with mathematics, going back thousands of years. Starting with the invention of zero, computing the value of pi, defining the trigonometric functions and computing their values for various angles, solving quadratic equations, giving rules for operations with negative numbers, computing the square roots of numbers, …; the breadth and depth covered are vast. Typically, results and formulas were expressed in compact, verse form.

In this short paper we look at a linguistic aspect of the work done in ancient India in trigonometry; namely, the names given to certain quantities. Specifically, we examine the relationship between Radius and Sine regarding their Hindi names. In Hindi, Radius is referred to as "त्रिज्या" and Sine is referred as "ज्या." The objective of this paper is to examine whether if there is any relationship between the Radius and the Sine values of some specific angles, त्रिज्या and ज्या, i.e., does the condition of 3 times sine (ज्या) equal to the radius (त्रिज्या) have any significance?

The word 'ज्या' means *chord* or *rope*. The word 'ज्या' is used in *Brhatsamhitā*, an encyclopaedic Sanskrit work written by Varāhamihira. We find references to this word in the *Shulba*

Keywords: Indian mathematicians, linguistics, exploration, trigonometry.

Sutras, which are among the oldest works written by ancient Indian mathematicians on geometry [1].

The Indian contribution to trigonometry was significant during the Gupta period and the work had relevance to astronomy. Aryabhata discovered the sine function and described the same in *Surya Siddhanta*. The three trigonometric functions studied by Aryabhata were the ज्या (sine), कोटि ज्या (cosine), and उत्क्रमज्या (tan) [3]. The Sanskrit word ज्या went through numerous adaptations and variations. After a few centuries, it ended up as *Sine* (a Latin word), the term in use today.

Discussion from perspective of Radius and Sine.

In modern mathematics, 'त्रिज्या' is referred to as radius and 'व्यास' is referred as the diameter. व्यास means *disjoined*, which breaks or distributes the circle into two halves (two semicircles).

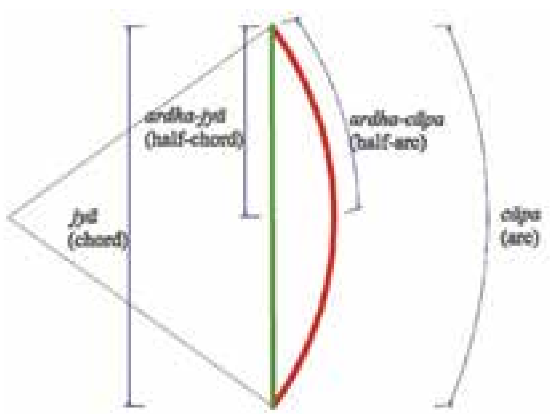


Figure 1

Referring to Figure 1, the arc of a circle is referred as धनु in Sanskrit. When the extremities of an arc are joined, like the string of the bow, a chord of the circle referred to as 'ज्या' is formed. Later, it was identified that the half chord or *ardha-jyā* is of greater relevance than the full chord [2]. Hence the qualifier *ardha* was omitted and 'ज्या' was used. Thus 'ज्या' gave reference of radius for computation purpose. The relationship between arc and the chord was defined by Bhaskara II as - *"What is really the arrow between the bow and bowstring is known as Versed sine."*

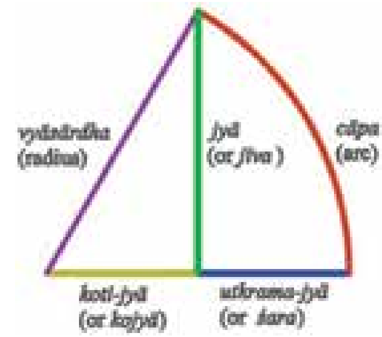


Figure 2

The arc of the circle looks like a bow and is called *dhanu*. As seen in Figure 2, when the arc of the circle 'धनु' subtends an angle of 90° at the center, it is called a quadrant of a circle or *vritta-pāda*. It is well known that there are 12 zodiac signs in astronomy; each zodiac sign defines an arc of 30°. According to Bhaskara I, "Three signs form a quadrant and these quadrants are distinguished as odd and even." The same was extended by Bhaskara II as – "Three signs form a quadrant and a circle is formed of four quadrants. These quadrants were again divided into odd *(ayugna, visama)* and even *(yugma, sama)*."

Interpreting the quotes of Bhaskara I and II, three consecutive zodiac signs define a quadrant or *vritta-pāda*. Further, the *ardha-jyā* of an arc of 90° in a circle is equal to the radius of the circle (since an arc of 180° corresponds to a semicircle, the corresponding chord is a diameter of the circle, which is twice the radius; hence the *ardha-jyā* corresponds to the radius). Keeping in mind that the qualifier *ardha* later got deleted, it makes sense that "the *jyā* of an arc of 90° in a circle is equal to the radius of the circle." Hence the term 'त्रिज्या' was coined by Hindu mathematicians to denote "the *jyā* of three signs," i.e., "the sine of three zodiac signs."

Note that this is only our conjectured explanation for the linguistic connection between these two terms. Though it seems very plausible, we may never know the full story.

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DR KOMAL ASRANI is currently serving as Professor in the Department of Computer Science and Engineering in BBDNIIT, Lucknow (India). She had obtained her Ph.D. in Computer Science Engineering. She also completed Diploma in Vedic Mathematics. Her areas of interest are Object Oriented Programming, Computer Graphics, Image Processing and Computation Analysis. She has published a number of papers in various International and National Journals. She has also conducted various Workshops for Vedic Mathematics. Currently, she is working to explore the application of Vedic Mathematics in Computer Science Engineering. She may be contacted at komalasrani74@gmail.com

INTERESTING NUMBERS

Solutions to questions on Interesting Numbers found on page 15

I. A few other four-digit numbers which exhibit the same property as 2210 are: **2205, 2704** and **5202**.

II. 15 + 16 + 17 + 18 = 66,

21 + 22 + 23 = 66.

66	×	2	=	132
66	×	3	=	198
66	×	4	=	264
66	×	5	=	330
66	×	6	=	396
66	×	7	=	462
66	×	9	=	594
66	×	10	=	660

132 (1 + 2 = 3, i.e., the sum of the leftmost digit and the rightmost digit in 132 = middle digit in 132)

(1 + 8 = 9, i.e., the sum of the leftmost digit and the rightmost digit in 198 = middle digit in 198)

(2 + 4 = 6, i.e., the sum of the leftmost digit and the rightmost digit in 264 = middle digit in 264)

III. $13^2 + 14^2 = 365$

IV. **32768** also exhibits the same property, (First digit – second digit + third digit)$^{\text{4th digit}}$ divided by 5^{th} digit, returns the same number. $\frac{(3-2+7)^6}{8} = 32768$

Wallace Jacob is a freelance writer who has written case-studies on Vijay Mallya, Chanda Kochhar and Tesla Inc. He has also authored five books. He may be contacted at wallace_jacob@rediffmail.com

एकवृन्तगतफलद्वयन्यायः। 'Two Fruits on One Stalk'

A S RAJAGOPALAN

एकवृन्तगतफलद्वयन्यायः is a popular maxim in Sanskrit literature used to describe situations when one effort gives two or many results simultaneously. Sometimes while solving problems, we stumble upon interesting results different from the intended results. One such example is given below.

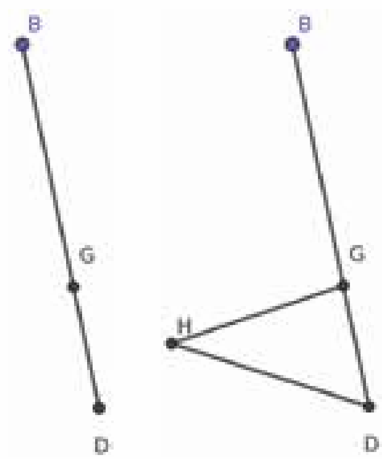
Problem: Using ruler and compass, construct the triangle whose median lengths are given.

Construction method

Let the given median lengths be represented by Ma (from vertex A), Mb (from vertex B), and Mc (from vertex C).

Step 1. Draw BD = Mb, and mark point G on BD at a length of $\frac{1}{3}$Mb from D.

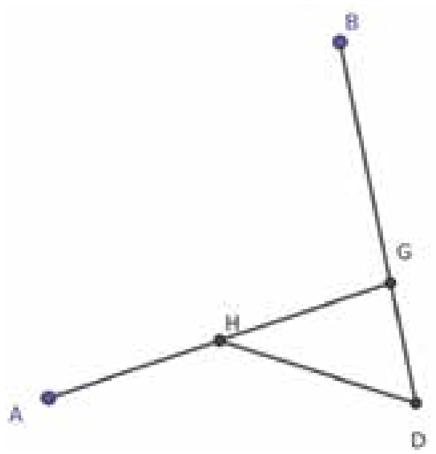
Step 2. Construct ΔGDH, with HG = $\frac{1}{3}$Ma and HD = $\frac{1}{3}$Mc. (Use standard construction methods to trisect segments Ma, Mb, and Mc.)



Step 1 Step 2

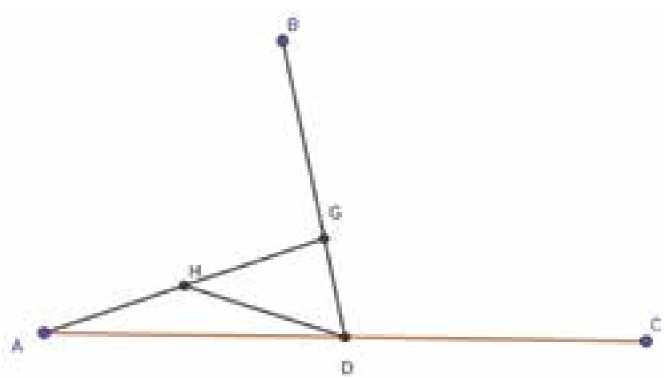
Keywords: Exploration, geometric construction, triangles, medians, areas.

Step 3. Extend GH to point A with GH = HA.



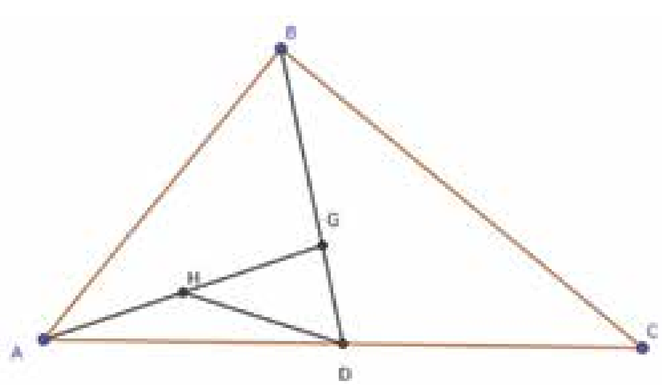
Step 3

Step 4. Construct segment AD and extend AD to point C with AD = DC.



Step 4

Step 5. Construct ΔABC. This triangle will have median lengths as Ma, Mb, and Mc.



Step 5

Analysis and justification

- Refer Figure 1.
- Let BD = Mb, AE = Ma, and CF = Mc be the medians of ΔABC to be constructed using compass and ruler. G is the centroid of ΔABC which divides the medians in the ratio of 2 : 1, the larger parts being towards the vertices of the triangle.
- Let H be the mid-point of AG. Since AG is $\frac{2}{3}$Ma, HG = $\frac{1}{3}$Ma.
- In ΔAGC, H and D being midpoints of sides AG and AC, HD will be half of GC and parallel to GC (Midpoint theorem).
- Since GC = $\frac{2}{3}$Mc and HD being half of GC, HD = $\frac{1}{3}$Mc.
- Now HD = $\frac{1}{3}$Mc, HG = $\frac{1}{3}$Ma and GD = $\frac{1}{3}$Mb.
- We started the construction with median BD and ΔGDH with HG = $\frac{1}{3}$Ma, HD = $\frac{1}{3}$Mc, and GD = $\frac{1}{3}$Mb.
- GH was extended to get point A (HA = GH) and AD was extended to get point C (AD = DC), thus completing the construction of ΔABC.

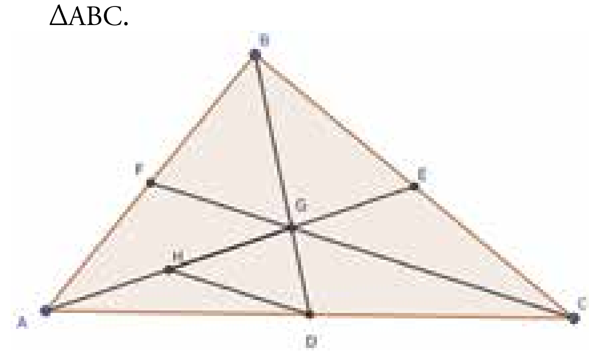


Figure 1

Note: Another interesting fact can be inferred from the above figure. ΔGDH has HG = $\frac{1}{3}$Ma, HD = $\frac{1}{3}$Mc and GD = $\frac{1}{3}$Mb. Therefore, ΔGDH is a scaled-down (similar triangle) version of the triangle formed by Ma, Mb, and Mc (medians of ΔABC) with a scale factor of 3 : 1.

If we take the area of ΔGDH to be 1 sq. unit, then the area of the triangle formed by Ma, Mb, and Mc (medians of ΔABC) will be 9 sq. units since these two triangles are similar with a scale factor of 3 : 1.

Since HG = HA, Area of ΔGDH = Area of ΔADH = 1 sq. unit (equal base and same height).

Area of ΔADG = Area of ΔGDH + Area of ΔADH = 2 sq. units.

Since AD = DC, Area of ΔADG = Area of ΔCDG = 2 sq. units (equal base and same height).

Area of ΔAGC = Area of ΔADG + Area of ΔCDG = 4 sq. units.

But Area of ΔAGC = $\frac{1}{3}$Area of ΔABC. Hence area of ΔABC = 3 times area of ΔAGC = 12 sq. units.

From the above it can be seen that area of the triangle formed by M_a, M_b, and M_c (medians of ΔABC) is 9 sq. units and area of ΔABC is 12 sq. units.

Therefore, the area of the triangle formed by the medians of a triangle is $\frac{3}{4}$ the area of the parent triangle.

A S RAJAGOPALAN has been teaching in Rishi Valley School KFI for the past 18 years. He teaches Mathematics as well as Sanskrit. Earlier, he was working as an engineer. He is keenly interested in teaching mathematics in an engaging way. He has a deep interest in classical Sanskrit literature. He enjoys long-distance running. He may be contacted at ayilamraj@gmail.com

Your challenge is to create a caption that crystallizes any of the mathematics in this picture! Send in your entries to AtRiA.editor@apu.edu.in

Haras Numbers

HARAGOPAL R

Definition: A Haras number is a n-digit number ($n \geq 3$) with the property that it is equal to sum of all the ($n-1$)-digit numbers whose digits are digits of the original number, *the circular wrap-around order being retained.*

For example, the 3-digit number $\boxed{abc}$ is a Haras number if $\boxed{abc} = \boxed{ab} + \boxed{bc} + \boxed{ca}$, i.e., if

$$100a + 10b + c = (10a + b) + (10b + c) + (10c + a),$$

which reduces to $100a + 10b + c = 11(a + b + c)$, and then to

$$89a = b + 10c.$$

Similarly, the 4-digit number $\boxed{abcd}$ is a Haras number if $\boxed{abcd} = \boxed{abc} + \boxed{bcd} + \boxed{cda} + \boxed{dab}$, i.e., if $10^3a + 10^2b + 10c + d$ is equal to

$$(100a + 10b + c) + (100b + 10c + d) + (100c + 10d + a) + (100d + 10a + b).$$

This reduces to $1000a + 100b + 10c + d = 111(a + b + c + d)$, and then to

$$889a = 11b + 101c + 110d.$$

Keywords: Circular wrap-around property

The 5-digit number $\boxed{abcde}$ is a Haras number if $\boxed{abcde} = \boxed{abcd} + \boxed{bcde} + \boxed{cdea} + \boxed{deab} + \boxed{eabc}$. This reduces to

$$10^4 a + 10^3 b + 10^2 c + 10d + e = 1111\,(a + b + c + d + e)\,,$$
$$\text{or: } 8889a = 111b + 1011c + 1101d + 1110e.$$

The general pattern may be seen from the above. The circular wrap-around property ensures that when we compute the sum of all the $(n-1)$-digit numbers whose digits are digits of the original number, each digit occurs with the same multiplicity.

More generally, the n-digit number $\boxed{a_{n-1}a_{n-2}\ldots a_2a_1a_0}$ is a Haras number if

$$10^{n-1}a_{n-1} + 10^{n-2}a_{n-2} + \ldots + 10a_1 + a_0 = 111\ldots1\,(a_{n-1} + a_{n-2} + \ldots + a_1 + a_0)\,,$$

where $111\ldots11$ has $(n-1)$ digits.

Let us now solve these equations and obtain some instances of such numbers. On solving the equations, we discover to our surprise that there only one such 3-digit number and only one such 4-digit number. Let us see why.

The case of 3-digit numbers.

The 3-digit number $\boxed{abc}$ is a Haras number if $89a = b + 10c$. We must find all possible solutions to this equation, a, b, c being digits with $a \neq 0$. Since b, c are digits, the quantity $b + 10c$ is at most 99, which implies that $a = 1$. Hence $b + 10c = 89$, and it is easy to deduce that $c = 8$ and $b = 9$. Hence the number is 198. (So, there is just one possibility.) We verify that it does satisfy the property:

$$\mathbf{198 = 19 + 98 + 81}.$$

The case of 4-digit numbers.

The 4-digit number $\boxed{abcd}$ is a Haras number if $1000a + 100b + 10c + d = 111(a + b + c + d)$. Note that this condition implies that $\boxed{abcd}$ is a multiple of 111 and therefore a multiple of 3. Hence the sum of the digits is a multiple of 3. This implies that $\boxed{abcd}$ is a multiple of 333 and therefore a multiple of 9. Hence the sum of the digits is a multiple of 9. This implies that $\boxed{abcd}$ is a multiple of 999. Hence $\boxed{abcd}$ is one of 1998, 2997, 3996, 4995, ... The sum of the digits of all these possibilities is 27 (all four-digit multiples of 999 have this form), so it follows that $\boxed{abcd} = 111 \times 27 = 2997$. (So, there is just one possibility.) We verify that it does satisfy the property:

$$\mathbf{2997 = 299 + 997 + 972 + 729}.$$

The case when the number of digits is 5 or more.

For larger numbers of digits, we may use similar reasoning (as we did above), but it is simpler to resort to a computer-assisted search. Here we report only the results.

- 5-digit numbers: there are just three such numbers: 13332, 26664, and 39996. Thus, we have

  $$\mathbf{39996 = 3999 + 9996 + 9963 + 9639 + 6399},$$

  and similarly for the others.

- 6-digit numbers: there is just one such number: 499995. Thus, we have:

$$\mathbf{499995 = 49999 + 99995 + 99954 + 99549 + 95499 + 54999}.$$

- 7-digit numbers: there is just one such number: 5999994.
- 8-digit numbers: there are just three such numbers: 23333331, 46666662, and 69999993.
- 9-digit numbers: there is just one such number: 799999992.
- 10-digit numbers: there is just one such number: 8999999991.

HARA GOPAL R works as a mathematics teacher in Municipal High School, Kurnool, Andhra Pradesh. He is very interested in finding new properties and relationships between numbers and geometrical figures. He may be contacted at rharagopal@gmail.com

Conversation on Tangram

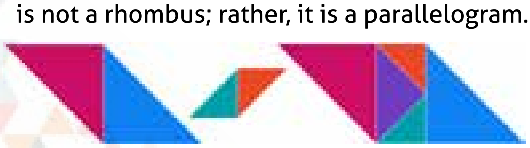
An email between Math Space and Khatri Jayeshkumar Nareshbhai, middle school mathematics teacher at Azim Premji School, Tonk, Rajasthan

Q: Can you make a rhombus that is not a square with the pieces of a tangram set?[i]

A: No, it is not possible to create a rhombus that is not a square using only tangram pieces. Although these pieces can be rearranged to form various shapes, including squares and parallelograms, they cannot be arranged to create a rhombus that is not a square. The angles of the tangram pieces are either right angles or 45°. Therefore, it is not possible to construct a rhombus that is not a square using a tangram set.

Q: Why can't one make a rhombus with 45° angles, i.e., rhombus with 45° and 135° angles?

A: When we tried to make rhombus with 45° and 135° angles as shown below, we found that it is not a rhombus; rather, it is a parallelogram.

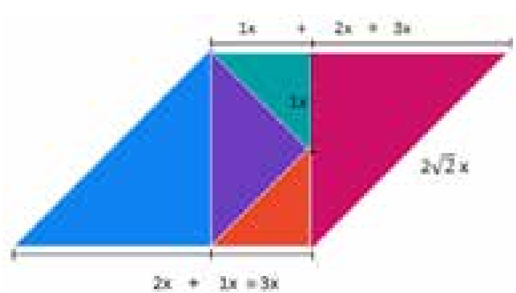


Q: One can easily see why the shapes made with 2 identical triangles are not rhombi – sides are 1 : √2. But the biggest one is almost a rhombus. So, why is it not a rhombus?

A: I think these images can prove that it is not a rhombus.

Q: Yes! 😊 But mathematically?

A:



[i] Tangram pieces consist of five triangles: two large triangles, one medium triangle, and two small triangles, along with a parallelogram and a square.

Irrational Nine-Point Centre is Impossible for a Triangle with Rational Vertices

SIDDHARTHA SANKAR CHATTOPADHYAY

In this short note, we prove that an irrational point in the Euclidean plane cannot be realized as the nine-point centre of a triangle all of whose vertices are rational points.

In the Iranian Mathematics competition at The University of Isfahan in March 1978, the following problem was given.

Problem 1. [1, 1.6.1] In the *xy*-plane a point is called 'rational' if both of its coordinates are rational.

Prove that if the centre of a given circle in the plane is not rational, then there are at most two rational points on the circle.

For the sake of convenience, we call a point 'irrational' if it is not rational as per the definition in Problem 1. The idea to solve Problem 1 is to assume, for the sake of contradiction, that there are three rational points on a circle whose centre is irrational and then arrive at a contradiction. Therefore, we can reformulate the above problem and assert that the circumcentre of a triangle is rational if the vertices of the triangle are all rational. In this article, we prove an analogous result for the centre of the nine-point circle, often referred to as the 'nine-point centre', for a triangle with rational vertices. The precise statement is as follows.

Keywords: Circumcircle, circumcenter, nine-point circle, nine-point centre, rational number, irrational number, rational point, irrational point

Theorem 1. Let P be an irrational point in the Euclidean plane. Then there does not exist any triangle ABC with all of A, B and C being rational points, such that P is the nine-point centre of $\triangle ABC$.

Before we proceed to prove Theorem 1, we recall that the nine-point circle of $\triangle ABC$ is the circle passing through the midpoints of the sides, the feet of perpendiculars drawn from the vertices to their respective opposite sides, the midpoints of the line segments joining the vertices to the orthocentre. Also, the nine-point centre is the midpoint of the line segment joining the circumcentre and the orthocentre of $\triangle ABC$.

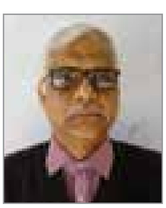
Proof of Theorem 1. Let $\triangle ABC$ be a triangle in the Euclidean plane such that A, B and C are all rational. Let the co-ordinates of A, B and C be (x_1, y_1), (x_2, y_2) and (x_3, y_3), respectively, where x_1, x_2, x_3, y_1, y_2 and y_3 are rational numbers.

We call a straight line $Ux + Vy = W$ 'rational' if U, V and W are rational numbers. Since B and C are rational points, the straight line joining these two points is rational. Also, the midpoint D of the line segment BC is a rational point because the co-ordinates are $\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$. Therefore, the perpendicular bisector of BC is a rational straight line. Similarly, the perpendicular bisector of AC is rational and thus the point of intersection of these two straight lines, i.e., the circumcentre of $\triangle ABC$, is a rational point.

Using the fact that BC is a rational straight line and A is a rational point, by a similar argument as above, we conclude that the straight line passing through A and perpendicular to BC is rational. Such is the case for the other two feet of perpendiculars drawn from the vertices B and C. Consequently, their point of intersection is a rational point. In other words, the orthocentre of $\triangle ABC$ is also a rational point. Therefore, the nine-point centre, which is the midpoint of the straight line segment joining the orthocentre and the circumcentre, is a rational point. □

References:

[1] B. R. Yahaghi, *Iranian Mathematics Competitions 1973–2007*, Hindustan Book Agency.



SIDDHARTHA SANKAR CHATTOPADHYAY is a retired Mathematics teacher from Bidhannagar Govt. High School, Salt Lake, Kolkata. He is actively associated with the Kolkata based 'Association for Improvement of Mathematics Teaching' for 30 years. He has participated in seminars and workshops for school students organized by Jagadis Bose National Science Talent Search over the past 15 years as a resource person, and has been engaged in spreading Mathematics Education across West Bengal by organizing seminars and workshops. He may be contacted at 1959ssc@gmail.com

Two 4-Digit Puzzles

ANAND PRAKASH

In this short article we pose and solve two playful problems about 4-digit numbers.

Notation. By $\boxed{abcd}$ we mean the 4-digit number whose digits are, respectively, a, b, c, d. So

$$\boxed{abcd} = 10^3 a + 10^2 b + 10c + d. \qquad (1)$$

Similarly for 3-digit numbers and 2-digit numbers; the meaning should be clear.

Problem 1. Find all 4-digit numbers $\boxed{abcd}$ with the property

$$a^2 + b^2 + c^2 + d^2 = \boxed{ab} + \boxed{cd}. \qquad (2)$$

Problem 2. Find all 4-digit numbers $\boxed{abcd}$ with the property

$$a^2 + b^2 + c^2 + d^2 = \boxed{abc} + d. \qquad (3)$$

Solution to Problem 1. Equation (2) may be rewritten as $a^2 + b^2 + c^2 + d^2 = 10a + b + 10c + d$, i.e., as

$$b(b-1) + d(d-1) = a(10-a) + c(10-a). \qquad (4)$$

Observe that (4) is symmetric in $\{a, c\}$ and also symmetric in $\{b, d\}$. Hence we may as well impose the additional restrictions $a \le c$ and $b \le d$. No essential loss results from these restrictions.

Since $b, d \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, it follows that $b(b-1),\ d(d-1) \in \{0, 2, 6, 12, 20, 30, 42, 56, 72\}$. Hence $b(b-1) + d(d-1)$ can assume only the values shown in Table 1.

Similarly, since $a, c \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, it follows that $a(10-a),\ c(10-c) \in \{9, 16, 21, 24, 25\}$. Hence $a(10-a) + c(10-c)$ can assume only the values shown in Table 2.

Keywords: Place value, constraints

0	2	4	6	8	12	14	18
20	22	24	26	30	32	36	40
42	44	48	50	54	56	58	60
62	68	72	74	76	78	84	86
92	98	102	112	114	128	144	

Table 1. Possible values of $b(b-1)+d(d-1)$ for $b, d \in \{0,1,2,\ldots,8,9\}$

18	25	30	32	33
34	37	40	41	42
45	46	48	49	50

Table 2. Possible values of $a(10-a)+c(10-c)$ for $a, c \in \{0,1,2,\ldots,8,9\}$

The numbers common to the two collections are the following:

$$18,\ 30,\ 32,\ 40,\ 42,\ 48,\ 50. \tag{5}$$

So to solve the stated problem, we must look for values of a, b, c, d such that

$$b(b-1)+d(d-1)=k=a(10-a)+c(10-c), \quad \text{where } k \in \{18,30,32,40,42,48,50\}. \tag{6}$$

We take up each k-value in turn.

- $\boxed{k=18}$
  We clearly have $(a,c)=(1,1)$ and $(b,d)=(3,4)$. Hence $\boxed{abcd}=1314$. (Of course, 1413 is also a solution but we choose not to list it.)
- $\boxed{k=30}$
  Here we have $a, c=(1,3)$ and $(b,d)=(1,6)$. Hence $\boxed{abcd}=1136$. (We do not list the permutations of this which are also solutions: 3116, 1631, 3611.)
- $\boxed{k=32}$
  Here we have $a, c \in \{2,8\}$ and $(b,d)=(2,6)$. Hence $\boxed{abcd}=2226$, 2286, or 8286. (As earlier, we do not list other solutions that are permutations of these.)
- $\boxed{k=40}$
  Here we have $a, c=(2,4)$ and $(b,d)=(5,5)$. Hence $\boxed{abcd}=2545$. (As earlier, we do not list other solutions that are permutations of these.)
- $\boxed{k=42}$
  Here we have $a, c \in \{3,7\}$ and $(b,d)=(1,7)$. Hence $\boxed{abcd}=3137$ or 3177.
- $\boxed{k=48}$
  Here we have $a, c \in \{4,6\}$ and $(b,d)=(3,7)$. Hence $\boxed{abcd}=4347$ or 4367.

- $\boxed{k = 50}$
  Here we have $a, c = (5, 5)$ and $(b, d) = (5, 6)$. Hence $\boxed{abcd} = 5556$. (As earlier, we do not list other solutions that are permutations of these.)

Hence the set of solutions is the following:

$$1314,\ 1136,\ 2226,\ 2286,\ 8286,\ 2545,\ 3137,\ 3177,\ 4347,\ 4367,\ 5556. \tag{7}$$

We have chosen not to list permutations of these solutions where $a > c$ or $b > d$ (though they may satisfy the stated condition). □

Solution to Problem 2. Equation (3) may be rewritten as $a^2 + b^2 + c^2 + d^2 = 100a + 10b + c + d$, i.e., as

$$a(100 - a) + b(10 - b) = c(c - 1) + d(d - 1). \tag{8}$$

The set of all numbers of the type $c(c - 1) + d(d - 1)$ for $c, d \in \{0, 1, 2, \ldots, 8, 9\}$ has been listed earlier, in Table 1. For convenience, we have listed the values again (Table 3).

The set of all numbers of the type $a(100 - a) + b(10 - b)$ for $a, b \in \{0, 1, 2, \ldots, 8, 9\}$ may be similarly computed; we have listed the values in Table 4.

It takes just a moment's glance to verify that this there are no numbers in common between the two collections.

It follows that Problem 2 has no solutions. □

0	2	4	6	8	12	14	18
20	22	24	26	30	32	36	40
42	44	48	50	54	56	58	60
62	68	72	74	76	78	84	86
92	98	102	112	114	128	144	

Table 3. Possible values of $c(c - 1) + d(d - 1)$ for $c, d \in \{0, 1, 2, \ldots, 8, 9\}$

108	115	120	123	124	205	212	217	220	221
300	307	312	315	316	393	400	405	408	409
484	491	496	499	500	573	580	585	588	589
660	667	672	675	676	745	752	757	760	761
828	835	840	843	844					

Table 4. Possible values of $a(100 - a) + b(10 - b)$ for $a, b \in \{0, 1, 2, \ldots, 8, 9\}$

ANAND PRAKASH runs a small garment shop at Kesariya village in the state of Bihar. He has a keen interest in number theory and recreational mathematics and has published many papers in international journals in these fields. He also has a deep interest in classical Indian music, poetry, and cooking. He has written a large number of poems in Hindi. He may be contacted at prakashanand805@gmail.com.

A design challenge from the tiling on the overbridge at Yadgir

Can your students re-create this tessellation and colour it differently? Send in your designs to AtRiA.editor@apu.edu.in

Congruency, A Trigonometric View

SHAILESH SHIRALI

In the article "Another Theorem for Congruence of Triangles" by Kasi Rao Jagathapu [1], published in *At Right Angles*, March 2023, the author points to the need for formulating additional congruence theorems. (See also [2].) It may be useful to look at this question from a trigonometric point of view.

Suppose we are given two sides of a triangle and one of the angles. Under what circumstances do these parameters uniquely fix the triangle? We consider the different possibilities.

Let the triangle be labelled ABC, and let the given sides be b and c. To fix the discussion, we assume that $b > c$. (We consider the much simpler case $b = c$ later.)

Suppose the angle specified is $\measuredangle A$ (the included angle). In this case we can calculate the third side a via the cosine rule ($a^2 = b^2 + c^2 - 2bc\cos A$). With three sides specified, the triangle is uniquely fixed. This obviously corresponds to the side-angle-side (SAS) congruence theorem.

Next, suppose the angle specified is $\measuredangle B$ (the non-included angle opposite the *longer* side). Since $c < b$, it follows that $\measuredangle C < \measuredangle B$ (strictly). Using the sine rule, we compute the value of $\sin C$:

$$\frac{\sin B}{b} = \frac{\sin C}{c}, \quad \therefore \ \sin C = \frac{c \sin B}{b}.$$

Knowing $\sin C$, we can determine a pair of supplementary angles whose sine is this value (recall that θ and $180^\circ - \theta$ have equal sines). One of these angles is obtuse and the other is acute. Angle C cannot be the obtuse angle, since $\measuredangle C < \measuredangle B$. *Therefore* $\measuredangle C$ *must be the acute angle*, which means that it is known. As we know both $\measuredangle B$ and $\measuredangle C$, we also know $\measuredangle A$, and therefore also side a; so the triangle is uniquely fixed.

Keywords: Congruency, side-angle-side (SAS) congruence, cosine rule, sine rule

To give a numerical example, suppose that $b = 5$, $c = 4$, and $\measuredangle B = 65°$. Then we have, from the above relationships:

$$\sin C = \frac{4 \cdot \sin 65°}{5} \approx 0.725,$$

so $\measuredangle C \approx 46.47°$ **or** $133.53°$. The latter option is not possible because we must have $\measuredangle C < \measuredangle B$; hence $\measuredangle C \approx 46.47°$. Therefore $\measuredangle A \approx 68.53°$, and the triangle is now determined fully. The situation has been sketched in Figure 1.

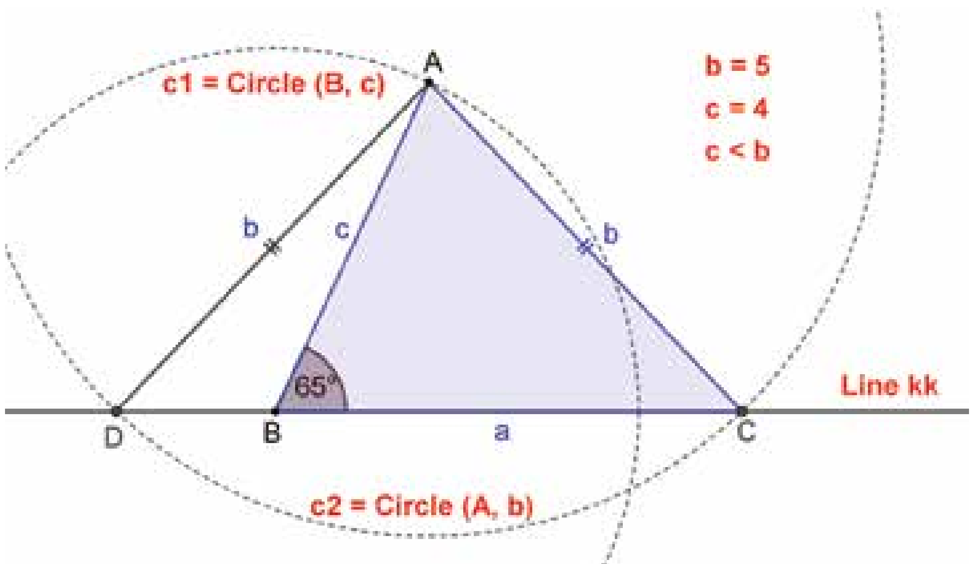


Figure 1. Construction of $\triangle ABC$ given $b, c, \measuredangle B$, with $b > c$

The steps of the construction are these:

1. Draw any arbitrary line kk and mark a point B on kk.
2. Draw a ray through B at the given angle $\measuredangle B$ with kk.
3. Draw a circle $c1$ with centre B and radius c. Let the ray through B meet $c1$ at A. This defines vertex A of the required triangle.
4. Draw a circle $c2$ with centre A and radius b. Since $b > c$, circle $c2$ will meet line kk at two points C and D, one on either side of B.
5. Let C be the point such that $\measuredangle ABC$ is equal to the given angle $\measuredangle B$. This defines vertex C of the required triangle, which is now fully determined.

It follows that if we know two sides of a triangle as well as the angle opposite the *longer* of the two sides, the triangle is uniquely fixed. This corresponds to the new congruency theorem proved in [1].

Now suppose the angle specified is $\measuredangle C$ (the non-included angle opposite the *shorter* side). As earlier, we are able to deduce the value of $\sin B$ via the sine rule, and thus restrict $\measuredangle B$ to two possibilities (a pair of supplementary angles). *But the logic used earlier to eliminate one of the two possibilities no longer applies*; we are not able to uniquely fix $\measuredangle B$. So we do not get any result here, and we do not obtain a congruence theorem.

To give a numerical example, suppose that $b = 4$, $c = 5$, and $\measuredangle B = 45°$. Then we have, from the above relationships:

$$\sin C = \frac{5 \cdot \sin 45°}{4} \approx 0.884,$$

so $\measuredangle C \approx 62.11°$ **or** $117.89°$. Now both options must be considered; neither one can be discarded. Accordingly, there are two possible triangles ($\triangle ABC$ and $\triangle ABC'$) with these specifications. See Figure 2. (The construction steps are the same as earlier.)

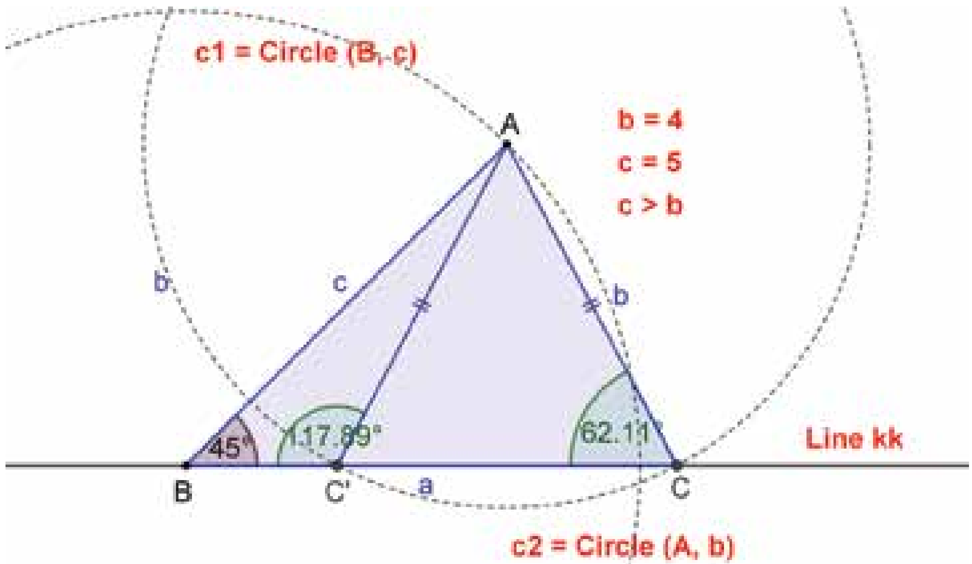


Figure 2. Construction of $\triangle ABC$ given b, c, $\measuredangle B$, with $c > b$

We round off the discussion by considering the case when the given sides are equal to each other. That is, we must construct triangle ABC given the sides b, c, with $b = c$, and the angle B. This is straightforward. For we have $\measuredangle B = \measuredangle C$ (since $b = c$), so $\measuredangle A - 180° - 2\measuredangle B$ is known, and therefore also side a, via either the sine rule or the cosine rule. Thus the triangle is fixed uniquely. Here too we obtain a congruency theorem.

Summarising, we may say that if we are required to construct triangle ABC given the sides b and c where $b \geq c$, and we are also given (the non-included) $\measuredangle B$, then the triangle is fixed uniquely.

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SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for *At Right Angles*. He may be contacted at shailesh.shirali@gmail.com.

Zeller's Congruence

ANUSHKA TONAPI

Historical Background

Zeller's Congruence is a method designed to find the day of the week corresponding to any given date in the Gregorian calendar. It was first discovered and proposed by Julius Christian Johannes Zeller, a German mathematician, and published in the journal of the Societe Mathematique de France.

Features of the Gregorian Calendar

In order to understand how exactly this method works, we must recognize which calendar we are applying it to and understand how the Gregorian calendar is designed. The Gregorian Calendar, introduced in 1582 by Pope Gregory as a modification of the Julian Calendar, is the universally accepted version. In this, one earth year consists not of 365.25 but 365.2425 days. To account for the decimal places, we add an additional day to the end of February every four years, since $0.25 \times 4 = 1$. Each such year is called a leap year and it has 366 days.

However, since $365.25 - 365.2425 = 0.0075$, adding one day in February once every four years will produce an over-estimate of $0.0075 \times 400 = 3$ days every 400 years. Thus, it was decided that the centurial years (years that mark the beginning of a century, such as 1800, 2000, etc.) shall be considered leap years only if they occurred every 400 years, that is, if they were divisible by 400. In any period of four consecutive centuries, there will be 24 leap years in the first three centuries and 25 in the fourth century (since the centurial year of that century is a leap year). Therefore, there will be $24 \times 3 + 25 = 97$ leap years for every four centuries in the Gregorian calendar.

Keywords: Calendars, days of week, pattern, cyclic property, modular arithmetic.

Since there are 97 leap years every four centuries, there is a total of $(365 \times 303) + (366 \times 97) = 146097$ days in four centuries. Since $146097 = 20871 \times 7$, there are 20871 weeks in four hundred years, and the next four centuries repeat a similar pattern. We call this the *cyclic property* of the Gregorian Calendar.

Cyclicity is a foundational feature of modular arithmetic. We will be looking at the use of modular arithmetic in finding the days of the week of any given date. This is also the reason why the formula is represented as a congruence and not a fixated equation.

We shall also be describing the *Doomsday Algorithm*, on the fiftieth anniversary of its development by British mathematician John Conway.

Finding the day of the week of any given date

This method consists of finding the first day of the year in which the required date is located and using it as a relation to the day of the week for our required date.

The first day of any given year

Assume for the sake of convenience that the year begins on March 1st, and the week starts with Monday $= 1$.

We first define a few variables. Suppose that:

- k represents the day of the month
- d_N represents the first day of March of a year N
- m represents the number of the month. Here, we have March = 1, April = 2, May = 3, and so on until January = 11 and February = 12.
- C represents the hundreds part of the year and Y represents the tens and units part of the year. For example, if $N = 1776$, then $C = 17$ and $Y = 76$.

The formula for d_N, where N=100C+Y, we have

$$d_N = \left(3 + 5C + Y + \left\lfloor \frac{C}{4} \right\rfloor + \left\lfloor \frac{Y}{4} \right\rfloor\right) \, mod\, 7$$

where the floor $\lfloor x \rfloor$ of a real number x is the greatest integer less than or equal to it.

For example, applying the formula to 1776 gives

$$3 + (17 \times 5) + 76 + \left\lfloor \frac{17}{4} \right\rfloor + \left\lfloor \frac{76}{4} \right\rfloor = 3 + 85 + 76 + 4 + 19 = 187 \equiv 5 \, mod\, 7.$$

Therefore, March 1, 1776, was a Friday.

Now that we know how to find the day of the week corresponding to March 1st of any year, we can find the first day of an arbitrary month in any given year.

At the beginning of month m, the day of the week shifts $\lfloor 2.6m - 0.2 \rfloor$ from March 1st of that year. This is represented by the formula $d_N + \lfloor 2.6m - 0.2 \rfloor - 2 \text{ mod } 7$.

For example, now that we know the day of the week of 1 March 1776, we can find the day of the week of 1 April 1776 using our formula:

$$d_{1776} + \lfloor 5.2 - 0.2 \rfloor - 2 = 5 + 5 - 2 = 8 \equiv 1 \bmod 7.$$

Thus, April 1, 1776 was a Monday.

Finding the day of the week of an arbitrary date

We shall use the first day of the given year to find the day of the week of our required date.

Let DW represent the required day of the week for an arbitrary date in the format DD/MM/YYYY. Let k be the day of the month in the year $N = 100C + Y$. Then, we have

$$DW = (k-1) + d_N + \lfloor 2.6m - 0.2 \rfloor - 2.$$

Substituting the formula for d_N gives

$$\begin{aligned} DW &= (k-1) + d_N + \lfloor 2.6m - 0.2 \rfloor - 2 \\ &= (k-1) + \left(3 + 5C + Y + \left\lfloor \frac{C}{4} \right\rfloor + \left\lfloor \frac{Y}{4} \right\rfloor\right) + \lfloor 2.6m - 0.2 \rfloor - 2 \\ &= \left(k + 5C + Y + \left\lfloor \frac{C}{4} \right\rfloor + \left\lfloor \frac{Y}{4} \right\rfloor + \lfloor 2.6m - 0.2 \rfloor\right) \pmod 7 . \end{aligned}$$

Example 1: We shall find the day of the week of the birthdate of the prominent Indian mathematician Srinivasa Ramanujan, which is 22nd December 1887.

We have $k = 22$, $m = 10$, $C = 18$, $Y = 87$. Applying our formula, we have:

$$\begin{aligned} DW = &\left(22 + (5 \times 18) + 87 + \left\lfloor \frac{18}{4} \right\rfloor + \left\lfloor \frac{87}{4} \right\rfloor + \lfloor 2.6 \times 10 - 0.2 \rfloor\right) \pmod 7 \\ &= (22 + 90 + 87 + 4 + 21 + 25) \pmod 7 \\ &= 249 \pmod 7 \equiv 4 \pmod 7 . \end{aligned}$$

Therefore, Srinivasa Ramanujan was born on a Thursday.

Example 2: Finding the day of the week on which India gained independence (15th August 1947).

We have $k = 15$, $m = 6$, $C = 19$, $Y = 47$. We apply our formula to get

$$\begin{aligned} DW = &\left(15 + (5 \times 19) + 47 + \left\lfloor \frac{19}{4} \right\rfloor + \left\lfloor \frac{47}{4} \right\rfloor + \lfloor 2.6 \times 6 - 0.2 \rfloor\right) \pmod 7 \\ &= 15 + 95 + 47 + 4 + 11 + 15 \pmod 7 \\ &= 187 \pmod 7 \equiv 5 \pmod 7 . \end{aligned}$$

Therefore, India gained independence on a Friday.

The Doomsday Algorithm (1973)

We close by describing John Conway's Doomsday Algorithm. Like Zeller's Congruence, it uses the periodic nature of the Gregorian calendar.

This algorithm is based on the observation that, in a year, certain dates fall on the same day of the week (trivially so if the gap between the two dates is an exact number of weeks, with no leap year complication). This verification may be done mentally without using complicated expressions and variables. This makes use of a 'landmark' date, called *Doomsday*, and knowing the day of the week of the Doomsday helps us find the day of the week of other dates in that year.

In order to find the day of the week of a given day, we need to know the Doomsday or the 'anchor day' of that year. Note that Doomsday 1900 was Wednesday, and that Doomsday increases by 1 every 12 years. The algorithm developed by Conway to find Doomsday of any given year in the 20th century is as follows:

(1) Find the quotient Q_1 when the last two digits of the year are divided by 12.

(2) Find the remainder R of the division in Step 1.

(3) Find the quotient Q_2 when this remainder (R) is divided by 4

(4) Add these three numbers to the Doomsday 1900 (Wednesday = 3) and take the final number modulo 7.

Consider odd-numbered and even-numbered months. Odd numbered months are January, March, May, July, September, and November, while the even numbered months are February, April, June, August, October, and December.

For even months, the n-th day of the nth numbered even month is Doomsday. That is, February 2nd, April 4th, June 6th, and so on until 12th December are all Doomsdays.

For odd-numbered months, the 9th day of the 5th month, the 5th day of the 9th month, the 11th day of the 7th month and the 7th day of the 11th month are all Doomsdays. (This can be memorized using the mnemonic "I work from 9 am to 5 pm at the 7-11", as knowing the above information requires the use of these 4 numbers in a mnemonic.)

Example: What day of the week was John Conway's birthdate?

John Conway was born on 26th December 1937. Applying the Doomsday Algorithm to 1937, we get the following. Dividing 37 by 12 gives quotient 3 and remainder 1, while dividing 1 by 4 gives a quotient of 0. Adding 3, 1 and 0 to Wednesday (Doomsday 1900, numbered 3) gives $3 + 1 + 0 + 3 = 7$, which taken modulo 7 is 0. Therefore, Doomsday 1937 is a Sunday.

According to our even month observations, 12th December 1937 was also Doomsday, and since the 26th December falls exactly two weeks from this date, it was a Sunday.

ANUSHKA TONAPI is a 9th grade student at Sri Kumaran Children's Home in Bangalore. She is a Math enthusiast with a strong interest in STEM subjects. She is also a creative writer and likes to write Poems and essays. She practices Carnatic vocal music as a hobby. She can be contacted at anushka.tonapi@gmail.com.

A Puzzling High School Math Problem

SOURAV DE

Introduction

High-school students may have seen problems related to infinite continued fractions such as:

Find the value of m, where,

$$m = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}.$$

Motivated by the repeating structure we substitute the value of m, in the right-hand side of the equation. So, we get

$$m = 1 + \frac{1}{1+m}, \quad \therefore m^2 - 2 = 0.$$

This tells us that either $m = \sqrt{2}$ or $m = -\sqrt{2}$. In the initial statement of the problem everything on the right-hand side of the equation was positive, so m must be positive. Hence, we discard the negative solution, and we get $m = \sqrt{2}$. Now let us look at a very similar problem.

The Problem

Find the value of x, where,

$$x = \cfrac{2}{3 - \cfrac{2}{3 - \cfrac{2}{3 - \cfrac{2}{3 - \dots}}}}.$$

Keywords: Infinite series, patterns, recurrence, algebra, quadratics.

First Attempt

Proceeding as before, we substitute the value of x in the right-hand side of the equation. Doing so we get

$$x = \frac{2}{3-x}, \quad \therefore x^2 - 3x + 2 = 0.$$

This tells us that either $x = 1$ or $x = 2$. There is no way to discard either of the solutions. So, we must take a different approach to solve the problem.

A Closer Look

Let us take a step back and think about the problem. What do the three dots at the end signify? What does it mean to find the 'value' of a continued fraction? Well, clearly the three dots at the end says that the pattern continues indefinitely. So, a continued fraction requires infinitely many mathematical symbols to be expressed.

In mathematics, the value of any expression which is not in "closed form" (an expression having finitely many mathematical symbols) is defined as follows: Define a sequence where each term is the expression after being chopped off at finitely many mathematical symbols, successive terms having an increasing number of mathematical symbols. The value that this sequence "approaches" (limit) is defined to be the value of the expression.

For example, the expression 0.1666… (which has infinitely many 6's) is defined as the limit of the sequence:

$$0, \quad 0.1, \quad 0.16, \quad 0.166, \quad 0.1666, \quad 0.16666, \quad \ldots$$

It is easy to see that this sequence approaches the value 1/6. So, we define $0.1666\ldots = 1/6$.

The value of an infinite continued fraction can be defined similarly, as the limit of the sequence formed by chopping the infinite continued fraction at regular intervals and evaluating the finite continued fractions. Any infinite continued fraction of the form

$$a_0 + \cfrac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \ldots}}}$$

can be chopped at regular intervals into finite continued fractions in two ways. This gives two sequences:

$$a_0, \quad a_0 + \frac{b_1}{a_1}, \quad a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2}}, \quad a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3}}}, \quad \ldots$$

and

$$a_0 + b_1, \quad a_0 + \frac{b_1}{a_1 + b_2}, \quad a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + b_3}}, \quad a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_4 + \ldots}}}, \quad \ldots$$

A Different Approach

Applying the same to the proposed problem, we get two sequences:

$$0, \quad \frac{2}{3}, \quad \frac{2}{3-\frac{2}{3}}, \quad \frac{2}{3-\frac{2}{3-\frac{2}{3}}}, \quad \ldots$$

and:

$$2, \quad \frac{2}{3-2}, \quad \frac{2}{3-\frac{2}{3-2}}, \quad \frac{2}{3-\frac{2}{3-\frac{2}{3-2}}}, \quad \ldots$$

The sequences can be simplified as:

$$0, \quad \frac{2}{3}, \quad \frac{6}{7}, \quad \frac{14}{15}, \quad \ldots$$

and

$$2, \quad 2, \quad 2, \quad 2, \quad \ldots$$

The first sequence clearly approaches 1, and the second sequence approaches 2. And we are stuck again. Can this problem be said to have a definite answer at all?

A sequence can be expressed as $\{t_n\}$, where t_n represents the n-th term of the sequence. Notice that, due to the repeating structure, both the sequences formed will follow a single (recurrence) relation expressing the $(n+1)$-th term of the sequence in terms of the previous terms.

Let $\{t_n\}$ be a sequence such that,

$$t_{n+1} = \frac{2}{3-t_n}.$$

We observe that $\{t_n\}$ describes the first sequence for $t_1 = 0$, and $\{t_n\}$ describes the second sequence for $t_1 = 2$.

Now, in some sense, the continued fraction is not just the two sequences formed; it represents this general recurrence relation. We can calculate a general expression for $\{t_n\}$ in terms of n and t_1 by solving the recurrence relation, after which we can evaluate the limit of $\{t_n\}$ for some general first t_1.

Solving the recurrence relation is slightly involved; it requires some concepts from the field of discrete mathematics (the 'characteristic equation') and is outside the scope of our discussion. After solving the recurrence relation, we get

$$t_n = \frac{2^n(2-t_1) + 4(t_1-1)}{2^n(2-t_1) + 2(t_1-1)}.$$

It is not too difficult to verify that this t_n satisfies the recurrence relation.

Now we evaluate the limit of $\{t_n\}$ as n goes to infinity. For all $t_1 \neq 2$, as n becomes large, the 2^n term dominates the constant term and the sequence $\{t_n\}$ approaches the value 1. For $t_1 = 2$, the sequence $\{t_n\}$ approaches the value 2.

We observe that, except a single "pole" at $t_1 = 2$, the recurrence relation describes a sequence which approaches 1 for all $t_1 \neq 2$. So, in our proposed problem we can assign $x = 1$ and discard the value $x = 2$.

Conclusion

A similar approach can be taken to find the value of any infinite continued fraction with repeating structure or one in which a recurrence relation can be established.

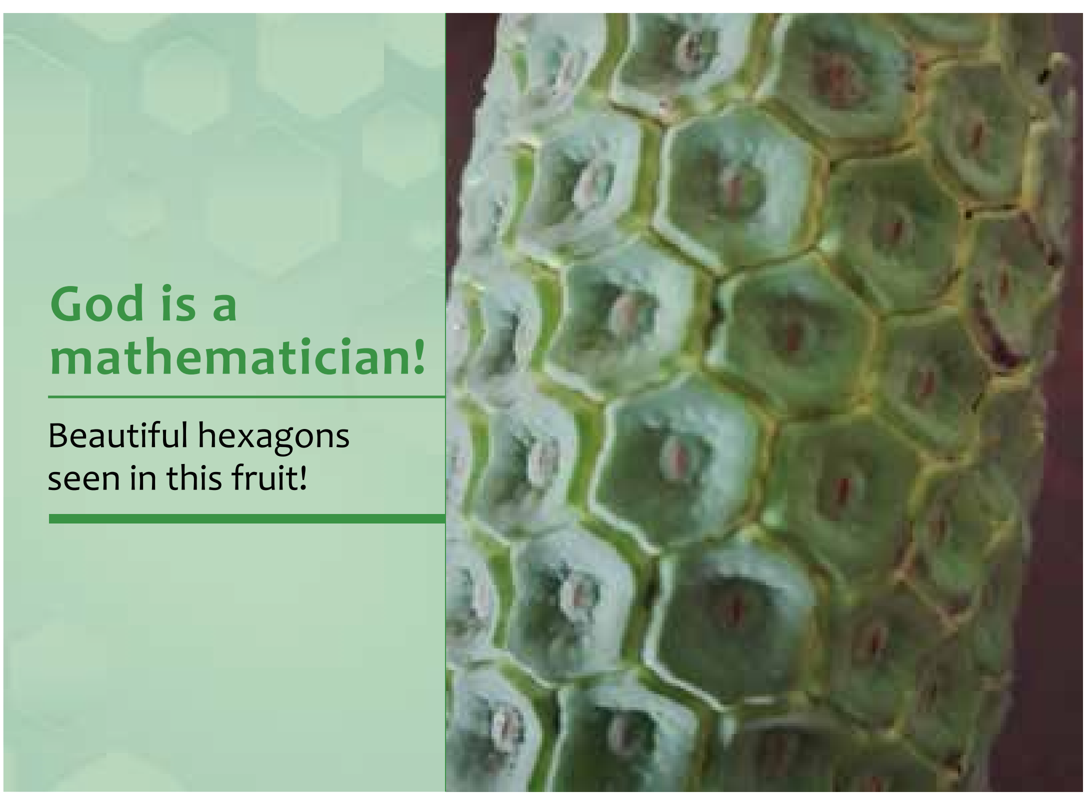
For example, we can solve the infinite continued fraction which I mentioned in the introduction using this approach. Again, from the repeating structure we observe that its two sequences formed by chopping the infinite continued fraction at regular intervals also follow a single recurrence relation,

$$c_{n+1} = 1 + \frac{1}{1 + c_n}.$$

The first and second sequences are obtained for $c_1 = 1$ and $c_1 = 2$ respectively. After finding the general expression for $\{c_n\}$ and evaluating the limit of $\{c_n\}$ as n goes to infinity, we find that for all $c_1 \neq -\sqrt{2}$, the sequence $\{c_n\}$ approaches the value $\sqrt{2}$. For $c_1 = -\sqrt{2}$, the sequence $\{c_n\}$ approaches the value $-\sqrt{2}$.

We observe that, except a single "pole" at $c_1 = -\sqrt{2}$, the recurrence relation describes a sequence which approaches $\sqrt{2}$ for all $c_1 \neq -\sqrt{2}$. So, in the problem we can assign $m = \sqrt{2}$ and discard the value $m = -\sqrt{2}$.

SOURAV DE is currently pursuing his B.Stat at Indian Statistical Institute, Kolkata. He is particularly interested in Euclidean geometry and Astronomy. He enjoys mathematical puzzles and games. One of his hobbies is making programmatic animations explaining/visualising mathematical concepts. Sourav may be contacted at desourav02@gmail.com.



A Problem from Madhava Mathematics Competition 2023

PARINITHA M

In this article, we discuss a solution to a number theory problem from Madhava Mathematics competition, 2023. (Madhava Mathematics Competition is a competition in mathematics for undergraduate students organized jointly by the Department of Mathematics, S.P. College, Pune, and Homi Bhabha Centre for Science Education, TIFR, Mumbai.)

Problem. Find all triplets (x, y, z) of non-negative integers satisfying the condition

$$x^2 + y^2 + z^2 = 16(x + y + z). \quad (1)$$

Solution. Let (x, y, z) be a triplet of non-negative integers satisfying condition (1). The condition implies that $x^2 + y^2 + z^2$ is even. Hence either two of x,y, z are odd and one is even, or all of them are even.

Since any odd square is of the form 1 (mod 4), if two of x, y, z are odd and one is even, then $x^2 + y^2 + z^2$ is of the form 2 (mod 4). But $16(x + y + z)$ is clearly a multiple of 4, so this possibility is ruled out. Hence all of x, y, z are even.

Let $(x, y, z) = 2(a, b, c)$ where a, b, c are non-negative integers. Substituting these into (1) we get $4a^2 + 4b^2 + 4c^2 = 32(a + b + c)$, hence

$$a^2 + b^2 + c^2 = 8(a + b + c). \quad (2)$$

Keywords: Parity, properties of square numbers, completing the square

The parity argument may be repeated and we see that all of a, b, c are even.

Let $(a, b, c) = 2(l, m, n)$ where l, m, n are non-negative integers. Substituting these into (2) we get

$$l^2 + m^2 + n^2 = 4(l + m + n). \qquad (3)$$

The parity argument may be used yet again and we see that all of l, m, n are even.

Let $(l, m, n) = 2(r, s, t)$ where r, s, t are non-negative integers. Substituting these into (3) we get

$$r^2 + s^2 + t^2 = 2(r + s + t). \qquad (4)$$

The parity argument may be used yet again and we see that all of r, s, t are even.

Let $(r, s, t) = 2(u, v, w)$ where u,v,w are non-negative integers. Substituting these into (4) we get

$$u^2 + v^2 + w^2 = 2(u + v + w). \qquad (5)$$

This may be written, by "completing the square," as

$$(u^2 - 2u + 1)+(v^2 - 2v + 1)+(w^2 - 2w + 1)=3, \qquad (6)$$

i.e., as $(u - 1)^2 + (v - 1)^2 + (w - 1)^2 = 3$.

Now, the only ways that 3 can be written as a sum of three squares are:

$$(\pm1)^2 + (\pm1)^2 + (\pm1)^2 = 3 \qquad (7)$$

It follows that

$$u-1 = \pm1,\ v-1 = \pm1,\ w-1 = \pm1, \qquad (8)$$

so $u = 0$ or 2; $v = 0$ or 2; $w = 0$ or 2. Any combination of these values leads to a solution of $u^2 + v^2 + w^2 = 2(u + v + w)$. Since $(x, y, z) = 8(u, v, w)$, we deduce that the solutions of the original equation are $x = 0$ or 16; $y = 0$ or 16; $z = 0$ or 16. Any combination of these values leads to a solution of (1).

Hence the solutions of the given equation are all the permutations of (0, 0, 0), (0, 0, 16), (0, 16, 16), and (16, 16, 16).

PARINITHA M is a II year BSMS student at IISER, Tirupathi. She is deeply interested in Mathematics, especially Analysis, Number Theory, and Group Theory. She took part in the Simon Marais Math Competition 2022 and the Madhava Mathematics Competition 2023. She is a recipient of the DST Inspire scholarship. She is an active member of The Math Club at IISER and hopes to pursue educational activities related to mathematics. Her interests include classical instrumental music. She may be contacted at parinitha.m.17@gmail.com

Explorations on the Sierpinski Gasket Graph

ANUSHKA TONAPI

In [6] the author describes the general properties of fractals and elaborates on the construction of a specific well-known fractal, the *Sierpinski Triangle*. The article delves into how pre-service teachers can explore various attributes of the fractal, such as the number of shaded triangles and shaded area at each stage. In this article we link the Sierpinski triangle (also known as the *Sierpinski Gasket*) to graph theory. We illustrate how the Sierpinski Gasket is related to the Sierpinski Gasket graph and explore Eulerian and Hamiltonian cycles in the graph using the CAS *Mathematica*.

Some mathematical preliminaries

In this section we shall elaborate on the mathematical concepts and definitions which are required for understanding the properties and characteristics of the Sierpinski Gasket graph.

The Sierpinski Gasket is a strictly self-similar fractal. This means that any arbitrary portion of the fractal at any given stage is a copy (at a reduced scale) of some previous stage of the fractal. It is initiated by considering a shaded or coloured equilateral triangular region whose sides are of unit length, and joining the midpoints of the three sides to create four smaller equilateral triangular regions. Once this is done, the central triangle is removed, leaving three smaller equilateral triangles. The initial equilateral triangular region is referred to as stage 0, and the next stage which comprises three scaled down smaller triangles (and a triangular hole) is stage 1 of the fractal. To continue the construction, the same process

Keywords: Fractals, Sierpinski, graph theory, Hamiltonian graphs, Mathematica.

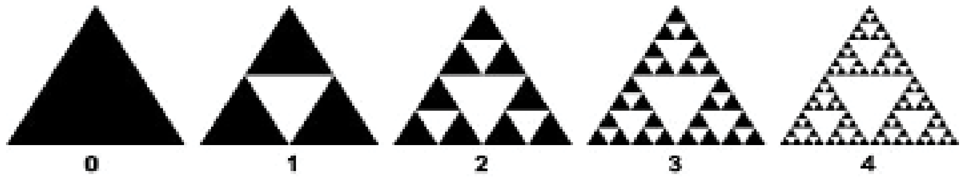


Figure 1. The first five stages of the Sierpinski gasket fractal. Image Credit: Fractal Foundation

is repeated on the three equilateral triangles of stage 1 to obtain stage 2 as shown in Figure 1. This process of replication continues wherein each smaller triangle is split into three still smaller triangles by joining the midpoints of these triangles and removing the central triangle. The reader may find the construction process in the article mentioned at the beginning of this article.

The next few definitions are related to graph theory, which is the study of *graphs*. In the context of graph theory, graphs are mathematical representations of pairwise relationships between objects.

A *graph* is a structure that consists of a set of *vertices* or *nodes*, denoted by *V*, and a set of edges, denoted by *E*, which consists of two-element subsets of *V*.

The number of vertices connected to a given vertex *v* through an edge is the *degree of that vertex* and is denoted by "deg" *(v)*.

A *walk* in a graph *G* is a finite sequence of consecutive edges and vertices that are all connected. A walk in which all the edges are distinct, that is, no edge is repeated, is called a *trail*.

A trail in which all vertices are distinct is called a *path*. If the walk ends at the same vertex where it began, then it is said to be closed. A closed path containing at least one edge is called a *cycle*.

A graph G is *connected* if there is a path between every pair of vertices in G.

A graph is *bipartite* if its vertices can be separated into two disjoint sets such that every edge in the graph connects a vertex in one set to a vertex in the other. However, there are no edges connecting pairs of vertices from the same set. An example of a bipartite graph is shown in Figure 2 where the blue and red vertices are disjoint sets and all the edges (the black lines) connect a blue vertex to a red one. However, there are no edges among the blue vertices nor the red ones.

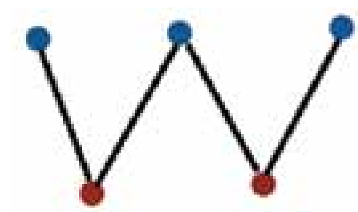


Figure 2. An example of a bipartite graph.

A connected graph G is *Eulerian* if there exists a closed trail containing every edge of G. Such a trail is called a *Eulerian trail*.

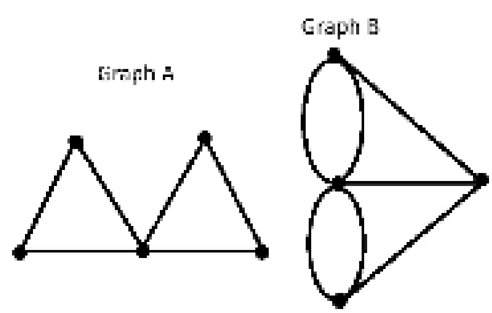


Figure 3. Graph A is Eulerian, while Graph B is non-Eulerian

A connected graph G is *Hamiltonian* if it contains a closed walk which passes through every vertex exactly once and in which no edge is repeated. Subsequently, a graph G is Hamiltonian if it contains a Hamiltonian cycle. Figure 4 is an example of a Hamiltonian cycle in a dodecagon graph.

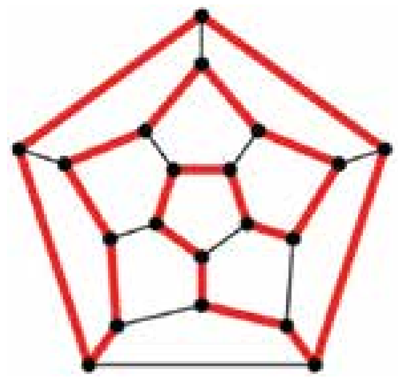


Figure 4. Hamiltonian Dodecagon graph.
Image credit: ResearchGate

The Sierpinski gasket graph and its properties

The *Sierpinski Gasket Graph* is the graph formed when the fractal of the same name is recognized as a system of edges and vertices rather than as a system of triangles. This is illustrated in Figure 5. It has the following properties: it is Eulerian, Hamiltonian, non-bipartite and connected. This means that it is possible to find a cycle which starts at a vertex, traverses each vertex and each edge exactly once. Further, it cannot be separated into two distinct sets of vertices A and B, such that all edges connect vertices in A with those in B.

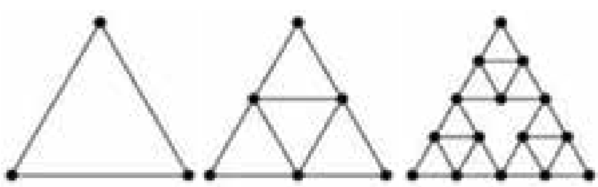


Figure 5. Stages 0, 1 and 2 of the Sierpinski gasket graph

Hamiltonian paths in the Sierpinski gasket graph

In this section we shall illustrate that the Sierpinski gasket graph is Hamiltonian, that is, it has a Hamiltonian path. A Hamiltonian path is a non-empty consecutive succession of edges that covers all the vertices in a graph exactly once. A Hamiltonian cycle is a closed walk in which all vertices in the graph appear exactly once and it ends at the same point at which it started. To determine whether a graph is Hamiltonian or not is a non-trivial problem.

Due to the Sierpinski graph's fractal nature and connectedness, there exists an inductive method of constructing Hamiltonian cycles in any given stage of the graph. For any stage n, we identify a Hamiltonian path and replicate this path in the three copies of the nth iteration that constitute the $(n + 1)$th iteration. This has been illustrated in Table 1 for $n = 0, 1, 2$ and 3, where all Hamiltonian paths (in the nth iteration) are indicated in blue and corresponding Hamiltonian cycles (in the $(n + 1)$th iteration) are represented in red.

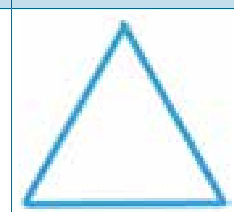
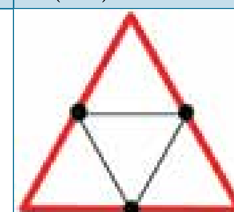
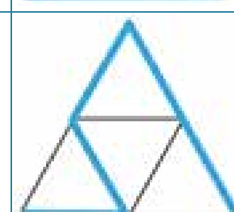
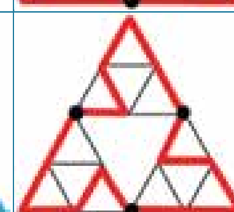
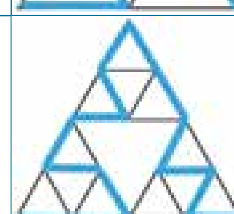
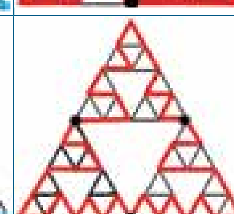
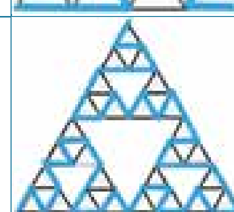
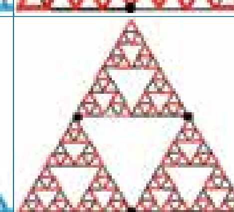
Iteration	Hamiltonian path in nth iteration	Hamiltonian cycle in $(n+1)$th iteration
$n = 0$		
$n = 1$		
$n = 2$		
$n = 3$		

Table 1.

We proceed by mathematical induction on n, or the nth iteration of the fractal graph. Consider the base case $n = 0$, the generator of the fractal graph, or the equilateral triangle from which the fractal is constructed. The generator itself is trivially Hamiltonian, since it is a closed loop where each of its three vertices is visited exactly once.

For our induction hypothesis, we shall assume that the Sierpinski Gasket graph of stage k is Hamiltonian, i.e., it contains a Hamiltonian path. We will prove that the Sierpinski Gasket graph of the $(k + 1)$th iteration is also Hamiltonian. Note that the $(k + 1)$th iteration of the graph is constructed using three replicas of the kth iteration. Before proceeding further, we need to introduce a nomenclature with regard to the vertices of the graph. The red vertices in figure 6 are referred to as *apex* vertices and the blue ones are referred to as *midpoint* vertices.

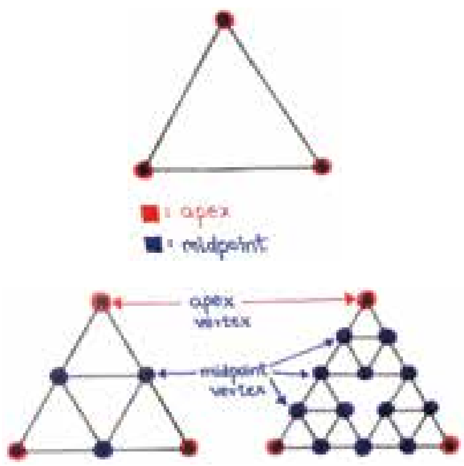


Figure 6. Apex and midpoint vertices of the Sierpinski Gasket graph

From our induction hypothesis, where we assumed that the kth iteration of the fractal graph is Hamiltonian, we can replicate the Hamiltonian path of the kth stage to traverse all three replicas of the (k + 1)th stage. Since the apex vertices of the three replicas in the (k + 1)th stage are connected through three edges (shown in black), we can traverse these edges to create a cycle which passes through all the vertices of the (k + 1)th stage exactly once. Therefore, this cycle is Hamiltonian by definition.

For example, the path in stage $n = 0$ can be reproduced using a similar method in its three replicas in stage $n = 1$ to produce a Hamiltonian cycle. This can be done by travelling the paths in the three replicas via the three connecting edges, as illustrated in the first row of Table 1.

Similarly, the path in stage $n = 1$ can be produced in the three replicas present in stage $n = 2$. These paths can be connected to make a Hamiltonian cycle via the three connecting midpoint vertices (marked as black dots) of these replicas.

However, owing to the graph's fractal nature, as n increases, the transition from the nth to the $(n + 1)$th stage becomes harder to construct. We will now discuss a method that can be used to construct Hamiltonian cycles on the Sierpinski gasket graph using Mathematica which can be accessed via the website wolframcloud.com.

Introduction to Mathematica

Wolfram Mathematica is a versatile software system based on the Wolfram language. It has powerful reach across various fields including mathematics, computer science, economics, statistics, machine learning, geometry and data science. It has numerous advantages which include a user-friendly programming interface, with a programming language built to resemble English-like function names and a coherent design. Its algorithms are very efficient, capable of tackling large-scale problems, due to its large database based on the Wolfram Knowledgebase.

In this section we will explore the graph theoretic features of the Sierpinski gasket graph with respect to programming components in the *Combinatorica* package of Wolfram Mathematica.

Constructing the Sierpinski Gasket Graph using Mathematica

In order to construct the graph, we need to use the built-in function in Mathematica called **`GraphData`**, which, depending on the type of command in the suffix, returns the required graph based on its conventional name, usually accompanied by its properties or its class. The Sierpinski gasket graph is obtained using the command **`GraphData [{"Sierpinski, n"}]`**. Here, n stands for the iteration number, and can range only between 0 and 5.

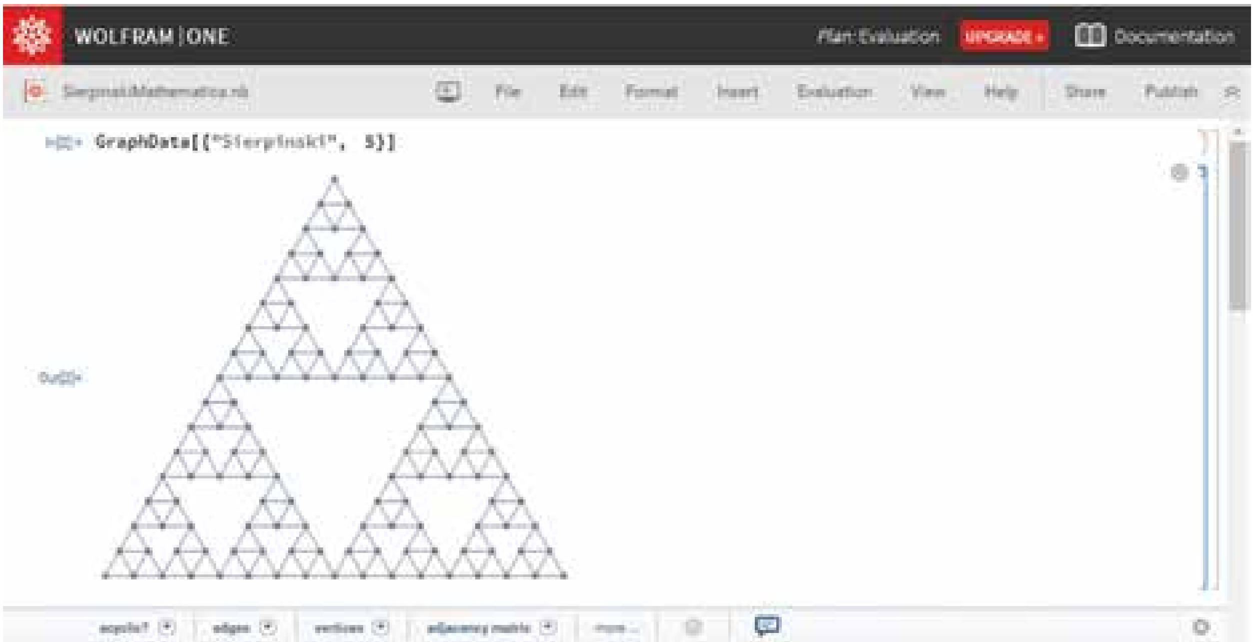


Figure 6. The fifth iteration of the Sierpinski gasket graph using Mathematica

For example, the 5th iteration of the Sierpinski gasket graph is generated in Mathematica as shown in Figure 6.

Constructing Hamiltonian Cycles using Mathematica

The **`FindHamiltonianCycle[g]`** function [4] helps us find a Hamiltonian cycle in the graph g, by giving us vertex directions with respect to labelled vertices in the graph. Mathematica chooses to label vertices using numbers. The **`HighlightGraph`** function helps us highlight the said cycle through a specific iteration of the graph. Figures 7 and 8 illustrate the Hamiltonian cycles in stages 1 and 2 of the Sierpinski gasket graph respectively.

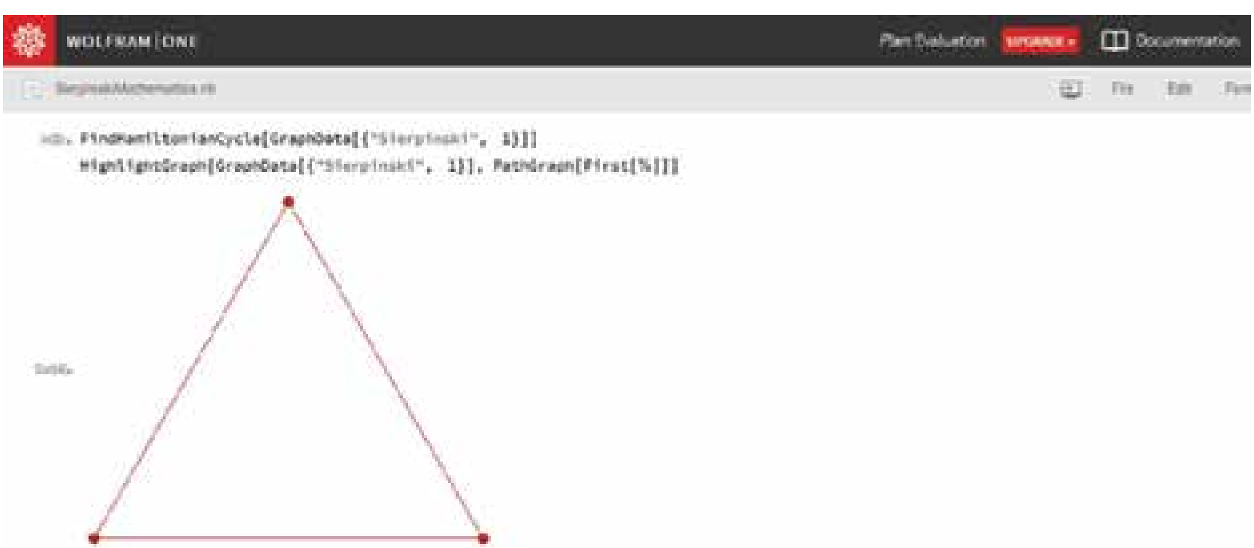


Figure 7. Hamiltonian cycle of stage 1 of the Sierpinski gasket graph

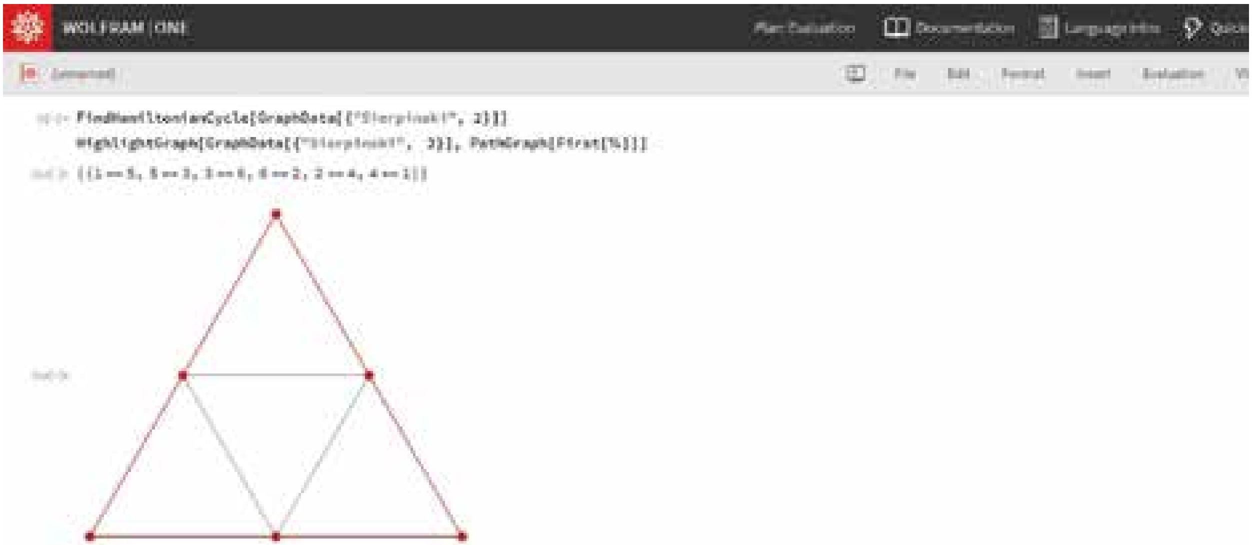


Figure 8. Hamiltonian cycle of stage 2 of the Sierpinski gasket graph

Figure 9. Hamiltonian cycle of stage 1 of the Sierpinski gasket graph

Figure 9 shows the output of the Mathematica codes when the functions are applied on the fifth iteration of the Sierpinski Gasket graph.

Eulerian Paths in the Sierpinski gasket graph

An Eulerian circuit is a closed walk in which each edge appears exactly once. It is derived from the Seven Bridges of Konigsberg Problem (which can be studied in detail in the article *Leonard Euler's Solution to the Konigsberg Bridge Problem*, by Teo Paoletti), wherein one must cross the seven bridges in a town without repeating a bridge. The bridges create a graph structure, and this problem was first solved by the mathematician Leonhard Euler.

Before exploring Eulerian paths, we need to mention an important result here.

A graph G is Eulerian if and only if every vertex in G is of even degree. An example is illustrated in the below graph.

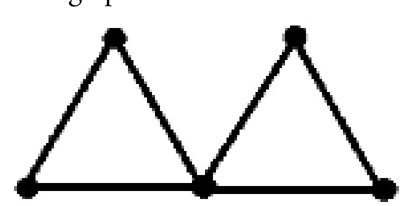


Figure 10. Example of an Eulerian graph in which every vertex is of even degree, that is, there are an even number of edges attached to every vertex.

An intuitive explanation for the even degree argument is that while travelling a graph, if it has an Eulerian cycle, we must enter every vertex via one edge and exit via another, i.e., there must be a receiving edge and a leaving edge. It can be shown that every vertex in the Sierpinski Gasket graph is of even degree, and hence the graph is Eulerian. Referring to figure 6, each apex vertex is of degree 2 since it is incident on only two other vertices. The midpoint vertices have degree 4, since they are incident on two vertices located in front of the vertex and two towards the right and left directions of that particular vertex. Therefore, all vertices in the Sierpinski Gasket graph have even degree.

Constructing Eulerian cycles using Mathematica

The `FindEulerianCycle[g]` function [5] helps us find an Eulerian cycle in the graph g, by giving us vertex directions with respect to labelled vertices in the graph. The `HighlightGraph` function helps us highlight the cycle through a specific iteration of the graph. The code (as shown in Figure 10) also displays the vertex progression of the cycle. The output figure of this code, unlike the Hamiltonian cycle code, does not highlight the edges since all the edges are covered in an Eulerian cycle. Thus, all the edges appear as one colour.

Real-world Applications

The Sierpinski Gasket graph has several applications. It is used in modelling quantum transport and quantum walks as discussed in [2]. It is used to investigate electronic properties and molecular chains to simulate experimental synthesised fractal nanostructures [3]. It also has many applications in the area of cellular automata.

It is modelled as a planar superconducting fractal lattice and exposed to a perpendicular magnetic field. The self-similarity of the fractal plays a role in addressing two central issues, namely, flux-quantization phenomena in loops and the low-field scaling behaviour of the magneto-inductance.

The Sierpinski triangle is used as a model for a bowtie antenna, showing advantages such as an efficient SERS substrate, 14% shrinkage (more compact) and higher resonance. The transducer of Sierpinski curve geometry was utilised for the miniaturisation of a microstrip patch strain sensor. The results showed the possibility of a dimension reduction due to the fractal structure.

Conclusion

In this article we have made connections between the Sierpinski gasket fractal and graph theory by exploring some properties of the Sierpinski Gasket graph. These properties elicit

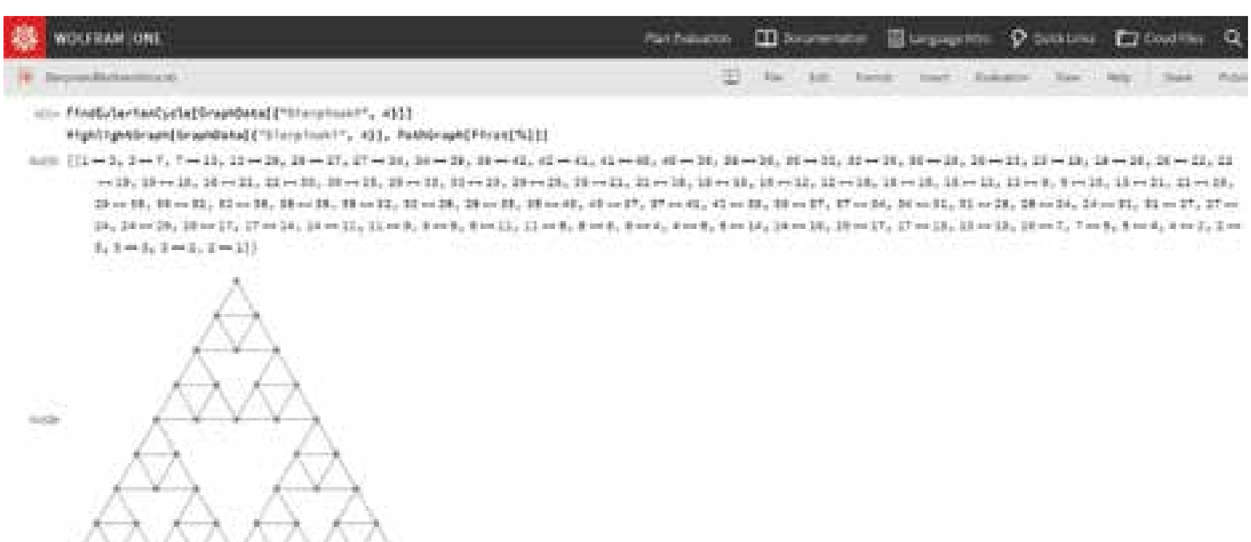


Figure 10: Generating an Eulerian cycle in the 4th iteration of the Sierpinski gasket graph.

the intricate nature of these fractal graphs. In particular, we have explored Eulerian and Hamiltonian cycles on the Sierpinski Gasket graph. However, such cycles tend to become intricate and complex as the number of stages of the fractal increases and it becomes difficult to construct these manually. Computer Algebra Systems such as Mathematica can be effectively used to identify such cycles. The results are faster and more accurate. Thus, this article illustrates the importance of computer algorithms with respect to cycle construction in graphs.

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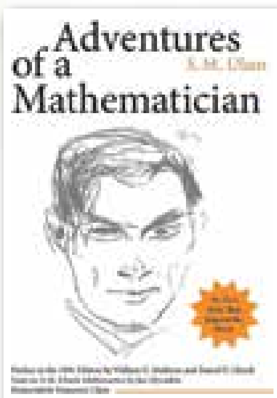
ANUSHKA TONAPI is a fourteen-year-old student of Sri Kumaran Children's Home in Bangalore. She is multifaceted, and loves to learn and discuss new things, especially about Mathematics, Science, app development, Carnatic music, and writing. She has published short stories and poems in various Children's magazines and newspapers. Anushka can be reached at anushka.tonapi@gmail.com

Adventures of a Mathematician By Stanislaw Ulam

Reviewed by Divakaran D



Stanislaw Ulam is probably best known among pure mathematicians for one of his early works with Karol Borsuk: the Borsuk-Ulam theorem. The special case of this theorem in dimension 2 is usually dubbed as "there are always two antipodal points on earth's surface that have the exact same temperatures and exact same pressures." Hidden behind is a reasonable assumption that temperature and pressure vary continuously on the surface of the earth. Among well-read non-mathematicians, he is probably well-known for his central role in the Manhattan Project. Few people would know that he invented the Monte-Carlo method, that he came up with the concept of cellular automata, and that he even made important contributions to the moon landing project and mathematical biology.

His intellectual life was truly an adventure making "Adventures of a Mathematician" an apt title for his autobiography.

Ulam lived through two World Wars and was directly impacted by them. During the first World War, he was still a child. But, Lwow, the city he lived in, was besieged by the Ukrainians in 1918. He recollects,

> *For the adults, it might have been a strenuous time, to say the least, but not for us. Strangely enough, my memories of these days are of the fun I had playing, hiding and learning card games with children for the two weeks before the siege was lifted with the arrival of another Polish army from France. This broke the ring of besiegers. For children, wartime memories are not always traumatic.*

Keywords: War, Stanislaw Ulam, Manhattan Project, fusion bomb, Feynman, Von Neumann, Fermi

In addition to the World Wars, his home was also affected during the Polish-Russian War in 1920. He lived through a time of great turmoil, especially for a Jew like himself. And the second World War affected him deeply. Although he was in USA by the time of the second World War, he knew Poland would no longer be the way he knew it to be and was deeply anxious for his family and friends back in Poland. A career in academics also seemed difficult during the beginning of the war as the celebrated mathematician Jacques Hadamard was offered a lectureship and Tarski was working as an instructor. Given these circumstances, he took up an instructorship in the University of Wisconsin Madison where he was later promoted to an Assistant Professor in his second year. Although he had a good time teaching and doing mathematics in Wisconsin, the war slowly directed his interests into mathematical physics and other applied areas. He also had a strong urge to help in the war effort. He even tried unsuccessfully to volunteer in the air force. It is this urge to contribute to the war effort that took him to Los Alamos and the work on the fission and fusion bombs.

In the book, Ulam does not talk a lot about his personal life, his emotions, or his feelings. In fact, we hardly get to know his wife Françoise. How he impressed his wife by sending her the letter d'Alembert sent to Mademoiselle de Lespinasse is among the few but very interesting aspects of his personal life that we get to know. This story was also interesting because the description of d'Alembert matched perfectly with Françoise Ulam's impressions of Ulam. A bit more of his personal life and self-reflections, I feel could have made the book even more interesting. The most personal/emotional he gets is when he describes his relationship with Von Neumann. We do get a sneak peek into the life of Von Neumann and his emotions behind certain things through the eyes of Ulam. Ulam also spends a chapter discussing the impact of the death of Von Neumann and Fermi, some discussions about death, the possibility of not being able to do mathematics in old age, etc. This is one of the most beautiful chapters in the book.

A lot of what is there in the previous paragraph stems from my preference for getting to know a person's thoughts, opinions, feelings, and emotions. Let me share some of the opinions or thoughts shared by Ulam that I found interesting. Ulam says that mathematicians tend to be vain and try to propose a linear order of "class" among well-known mathematicians. He also freely shares his ordering and judgement on various famous mathematicians and physicists. These subjective judgments often, but not always, match their fame and are very interesting. Being an educator, I was also interested in his thoughts about mathematics education. He says,

> *One may wonder whether teaching mathematics really makes much sense. If one has to explain things repeatedly to somebody and assist him constantly, chances are he is not cut out to do much in mathematics. On the other hand, if a student is good, he does not really need a teacher except as a model and perhaps to influence his tastes.*

Interestingly, this matches a lot with what Feynman says about teaching, quoting Gibbon:

> *the power of instruction is seldom of much efficacy except in those happy dispositions where it is almost superfluous.*

These thoughts are somewhat disturbing for an educator. I try to come to terms with it by thinking that this probably might be the case for the few who will rise to the absolute top. But the vast majority of people need not do "much" (at least in Ulam's eyes) in mathematics. I believe that good teaching can make a significant impact on students who are trying to acquire a basic understanding of mathematics which they may use in other spheres of life. Moreover, I also believe that many mediocre researchers would not have become what they have if not for the help of a dedicated teacher.

He also ponders what leads to some being a lot more successful than others. He believes an ability to imitate in childhood and some inborn curiosity in the field is certainly important. But,

> *Another determining factor may be the initial accidents of success or failure in a new pursuit. I believe that the quality of memory develops similarly as a result of initial accidents, random external influences, or a lucky combinations of the two. Nothing succeeds like success, it is well-known, especially in early youth.*

As Ulam had practised mathematics in multiple languages, he shares his impressions of the impact of language on the way mathematics is done. He also compares the difference in cultures of mathematics in various countries—often emphasising the strengths and weaknesses of each country. To give an example, he says,

> *I had my meals at Adams House, and the lunches were particularly agreeable. We sat at a long table — young men and sometimes great professors; the conversations were very pleasant. But often, towards the end of a meal, one after the other would gulp his coffee and suddenly announce "Excuse me, I've got to go to work!" Young as I was I could not understand why people wanted to show themselves to be such hard workers.*

> *In Poland, people would also pretend and fabricate stories, but in the opposite sense. They might have been working frantically all night, but they pretend they never worked at all. This respect for work appeared to me as a part of the Puritan emphasis on action versus thought, so different from the aristocratic traditions of Cambridge, England, for example.*

Ulam was most famous for asking excellent questions and also guessing the answers to some of them. True to his nature, the book also contains many scientific and mathematical questions waiting to be answered. One such question, called the Ulam game or Renyi-Ulam game has gained widespread popularity. There are several other questions like that and one of my favourites is the following:

> *As is well known, the theory of special relativity postulates and is built entirely on the fact that light always has the same velocity regardless of the motion of the source or the observer. From this postulate alone everything follows, including the famous formula $E = mc^2$. Mathematically speaking, the invariance of the cones of light lead to the Lorentz group of transformations. Now a mathematician could, just for mathematical fun, postulate that the frequency, for example, remains the same, or that some other class of simple physical relation is invariant. By following logically, one could see what the consequences would be in such a picture of a not "real" universe.*

Finally, let me get to an aspect of his life that may be a bit controversial—his role in the development of the fission and fusion bombs and his thoughts about the same. During the development of the fission bomb during the second world war, he was actively looking to help the war effort. About returning to Los Alamos to work on the fusion bomb, he says,

> *I felt no qualms about returning to the laboratory to contribute further studies of the development of atomic bombs. I would describe myself as having taken a middle course between naive idealism and extreme jingoism. I followed my instincts (or perhaps the lack of instincts) and was mainly interested in the scientific aspects of the work. The problems of nuclear physics were very interesting and led into new regions of physics and astrophysics. Perhaps I also felt that the technological sequels to scientific discoveries were inevitable.*

I am however not convinced that he was mainly interested in the scientific aspects. I believe he trusted his new home and thought that if anyone should hold such power, it should be the USA. I feel so, because a few lines before, he says, *I was in favour of continuing strong armament policies if only not to run the risk of being overtaken by other nations.*

In general, I believe that, even for the most rational people, many of their decisions are guided by emotions—they are just better at rationalising the decision post facto. He was a victim of many wars and lost most of his family to the second World War. The following words of Françoise Ulam suggest that he probably believed these bombs will be a deterrent for future wars.

When I voiced my reservations about still living at the heart of the thermonuclear work, Stan would reassure me that barring accidents, the H-bomb rendered war impossible. He also agreed, however, that there are too many bombs already, and he did not believe that Russia would invade Western Europe, one of the supposed reasons for super-rearmament.

But, sadly we may never know his true thoughts, as his reflections in the book may not be completely reliable. In 1945-1946 he had to undergo brain surgery. He says,

By the way, many of the recollections of what preceded my operations are hazy. Thanks to what Françoise told me later I was able to put it together.

Although we have an unreliable narrator, it cannot be far away from reality, else readers would have pointed out inconsistencies between his version and reality. So, the unreliability of the narrator only puts into doubt the motives or emotions behind real incidents. Perhaps this is why there is so little about emotions. Of course, I am letting my imagination run loose and this is certainly just my reading. Please read through it and let me know what you feel.

To summarise, the life of Stanislaw Ulam is a window into the troubling political scenario and the exciting scientific and technological developments of the time. The book would be of great interest to those who are interested in the history of science or/and are interested in understanding what transpires in the minds of scientists and mathematicians as they change the course of human affairs "through a few scribbles on a blackboard or on a sheet of paper". The life of Ulam transcends the separation of mathematics into "pure" and "applied" and gives a lot of insights into the culture of doing mathematics. Thus, it would be of great interest to mathematicians and mathematics educators. However, I feel, it might not be suitable to those who are starting their journey in the world of mathematics. Ulam fits well into the stereotype of a genius with innate potential. His life and some of his opinions may discourage some of the young learners who already doubt if they are "worthy" of being a mathematician. I believe everyone is worthy and we can all make contributions to the best of our abilities— like the squirrel that helped Rama build his bridge!

DIVAKARAN D works at Azim Premji University, where his main focus is undergraduate education. He has a deep interest in pedagogy. His research interests are in the fields of Geometry and Topology. He obtained his PhD from the Indian Institute of Science, Bangalore, and followed this with post-doctoral fellowships at the Institute of Mathematical Sciences, Chennai, at the University of Edinburgh, and at IISER Bhopal. Most recently he has been working on the Lean Theorem Prover. He may be contacted at divakaran.d@azimpremjifoundation.org.

Centroids of Quadrilaterals and a Peculiarity of Parallelograms

HANS HUMENBERGER

Dedicated to Arnold Kirsch (Germany, 1922-2013) on the 10^{th} anniversary of his death.

Abstract: We analyze briefly different kinds of centroids of quadrilaterals and give geometrical and elementary proofs that in the world of quadrilaterals, *only parallelograms* have the property that their *laminar centroid* coincides with the *vertex centroid*. This paper is based on short papers (in German) by Arnold Kirsch (Kassel, Germany, 1922-2013) published between 1987 and 1995. We think these deserve to be better known – published proofs in mathematical journals in English language are usually rather complex (e.g., Kim 2016, 2020).

Definitions. A *lamina* is a flat object of uniform thickness. The *laminar centroid* of a flat region is the centre of gravity of the region when it is regarded as a thin lamina. It is also called the *geometric centroid*. The *vertex centroid* of a polygon is the centre of gravity of a system of unit masses placed at the vertices of the polygon.

In the world of triangles, the *laminar centroid* always coincides with the *vertex centroid* (intersection point of the medians). This is an elementary and well-known fact. We proceed to prove this using the principle of levers from elementary physics.

Keywords: Geometry, centroids, parallelograms, laminar centroid, vertex centroid

Lemma 1: The vertex centroid G of a pair of point masses (weights $w_1$and w_2) lies on the connecting line and the corresponding distances l_1, l_2 have the ratio $\frac{l_1}{l_2} = \frac{w_2}{w_1}$. For mechanical purposes, one can imagine that at the centroid G, a combined weight $w_1 + w_2$ is concentrated. In terms of analytical geometry, the point G is the *weighted arithmetic mean* of the points G_1 and G_2:

$$G = \frac{w_1}{w_1 + w_2} \cdot G_1 + \frac{w_2}{w_1 + w_2} \cdot G_2.$$

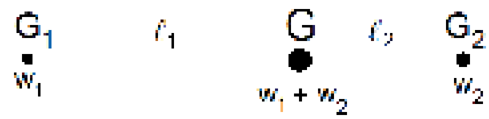


Figure 1. Law of levers.

Drawing on this fact, one can give a physically motivated proof that the medians of a triangle concur, intersecting each other in the ratio 2: 1. Assume that at the vertices of a triangle we have unit point masses, and we want to determine the centroid of these three point masses (we call this the 'vertex centroid'). Lemma 1 tells us that the centroid of the pair of unit masses at A and B is at the midpoint M_{AB} of AB, where we then have mass 2. Using Lemma 1 again, we see that the centroid of the three unit-masses lies on the median $m_c = CM_{AB}$, at the point G such that CG: $GM_{AB} = 2$: 1. We may imagine all three unit-masses to be concentrated at G (with total mass 3).

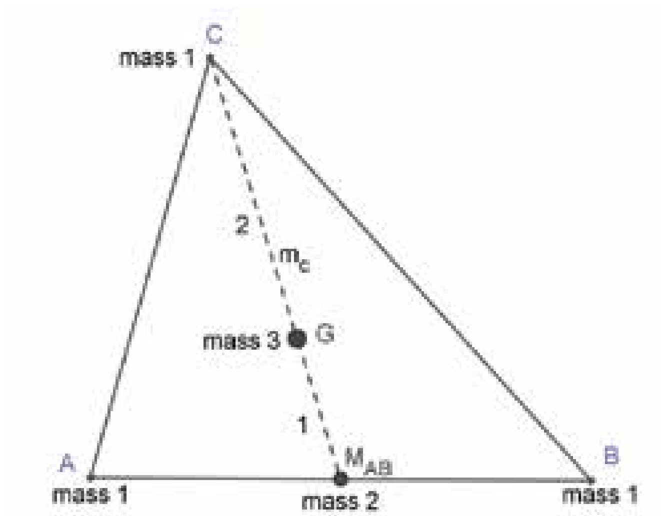


Figure 2. The triangle centroid as the *vertex centroid.*

Since the same must hold for the other two medians, and the centroid is unique, we have proven two things: (1) The medians concur at a point that trisects all three of them; and (2) the point of concurrence is the *vertex centroid*. Note that this approach does not explain why the laminar centroid of the triangle lies at the same point. Here is one approach which explains why. We divide the triangle into infinitesimally thin stripes parallel to AB. Each of these stripes has its center of mass at its midpoint, so the center of mass of the whole lamina must lie somewhere on the line consisting of all these midpoints, which is the median CM_{AB}. By a symmetric argument, it must also lie on the other two medians, hence the intersection point of the three medians is also the laminar centroid.

Let us denote the *laminar centroid* of a polygon by G_L and its *vertex centroid* by G_V.

The following must be noted. The property $G_L = G_V$ is a peculiarity of triangles (in the sense that it is true for *all* triangles), but for other polygons this is not necessarily true. Of course, for regular polygons $G_L = G_V$ still holds (by symmetry, both must lie at the centre of the polygon), but for general polygons it is of great interest to ask: For which polygons is it true that $G_L = G_V$? We will restrict our exploration in this only article to *quadrilaterals* and ask: Which quadrilaterals have the property $G_L = G_V$? (We are not aware if there are any results of this kind for polygons with more than 4 vertices.)

It is easy to see that *parallelograms* have the property that $G_L = G_V$ (= intersection point of the two diagonals). Assume we have a parallelogram *ABCD* with unit mass at each vertex. Then according to Lemma 1, the intersection point of the diagonals (where they bisect each other) is the centroid of the two masses at *A* and *C* (mass 2 units), and also of the two masses at *B* and *D* (again mass 2 units). Hence the point of intersection of the diagonals is the vertex centroid G_V.

To see that this is G_L too, we consider the laminar centroids of triangles *ABC* and *ADC*, namely, $G_L(ABC)$ and $G_L(ADC)$. (These coincide with respective vertex centroids.) Both lie on the diagonal *BC* and lie at equal distance from the point of intersection of the diagonals (note the half-turn symmetry of a parallelogram with this point as centre). For mechanical purposes we can imagine the whole masses of triangles *ABC* and *ADC* being concentrated at these two triangle laminar centroids. And since these masses (areas) are equal, it follows that the laminar centroid of the whole parallelogram lies at the intersection point of the diagonals (Figure 3). So, all parallelograms have the property that $G_L = G_V$.

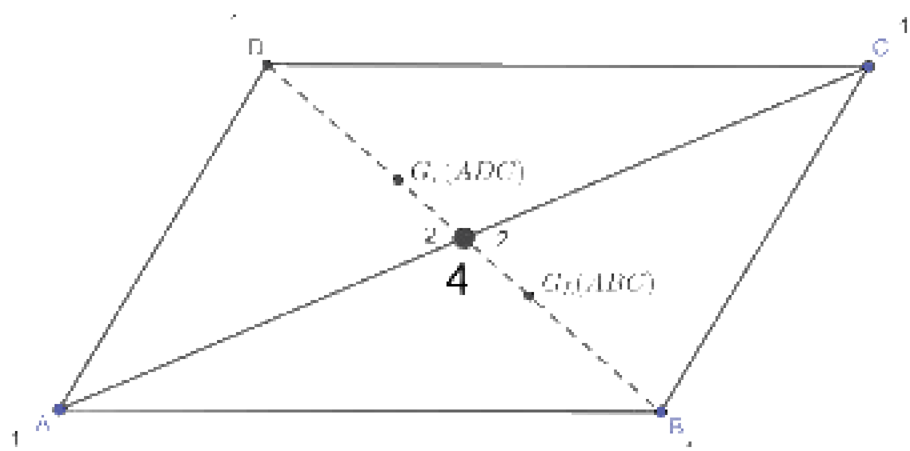


Figure 3. The intersection point of the diagonals of a parallelogram are G_V and G_L.

What has been proved above is well known. Now for the not so well-known part: *parallelograms are the only quadrilaterals with the property* $G_L = G_V$. Here Arnold Kirsch (of Germany, a very deserving professor for mathematics and mathematics education at the University of Kassel) came up with a very elementary proof which students in grades 9 or 10 can follow and which not only answers the question (*verification*) but explains *why* it is true (*explanation*). *Verification* and *explanation* are two important functions of proof (but there are also others, see De Villiers 2012).

We did not find an elementary proof in the English literature (if somebody happens to know one, please inform the author), hence we wanted to share his ingenious ideas, formulated in German (Kirsch 1987, 1995), with potentially more readers in the English language.

Theorem: A quadrilateral has the property that $G_L = G_V$ if and only if it is a parallelogram.

For this topic, we can omit crossed quadrilaterals because it is not so clear what is the interior of such a quadrilateral, and we restrict to convex or concave quadrilaterals (Figure 4a, 4b). In both cases there is an interior diagonal (*AC* in Fig. 4) which itself or its extension meets the other diagonal (*BD* in Fig. 4).

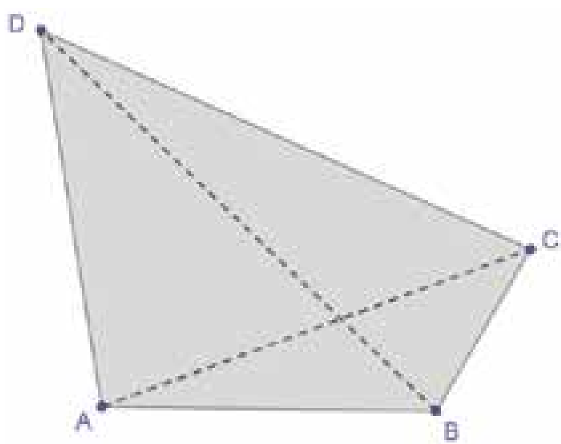


Figure 4a. Convex quadrilateral.

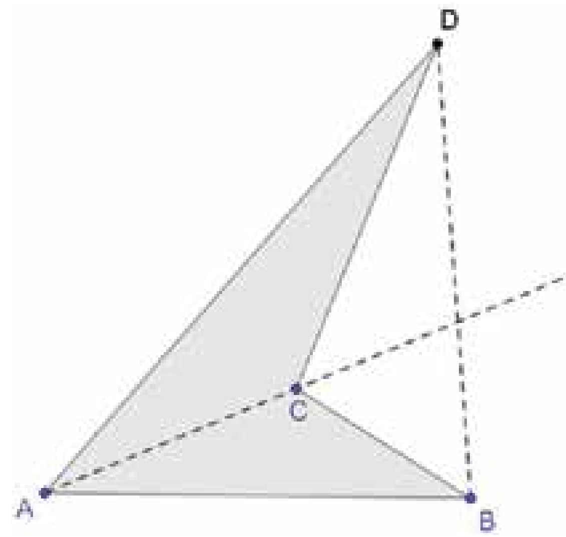


Figure 4b. Concave quadrilateral.

The part "if" of the Theorem (the easy and well-known part) has been dealt with above. For the "only if" part we use another lemma. We lay the groundwork for this by stating two facts:

(a) If the vertex D of ΔDAC is moved parallel to AC by $\boldsymbol{u}$, then the centroid G_2 of ΔDAC is moved by $\frac{1}{3}\boldsymbol{u}$ (see Figure 5; here M denotes the midpoint of AC). This should be clear since $G_2 = \frac{D+A+C}{3}$.

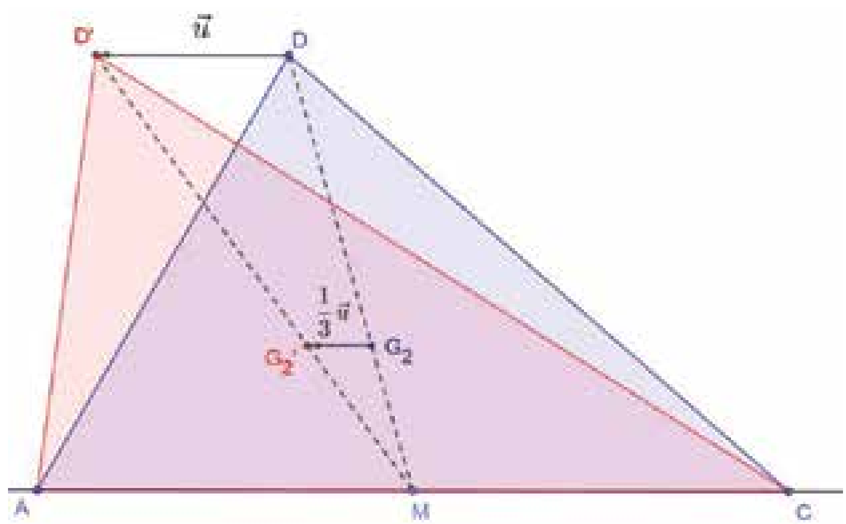


Figure 5. If $D' = D + \boldsymbol{u}$, then $G_2' = G_2 + \frac{1}{3}\boldsymbol{u}$.

(b) If vertices A and C of ΔABC are moved along the straight line AC by $\boldsymbol{v}$, then the centroid G_1 of $\Delta ABC$is moved by $\frac{2}{3}\boldsymbol{v}$ (see Figure 6; M is the midpoint of AC; M' is the midpoint of $A'C'$). This is so since $G_1 = \frac{A+B+C}{3}$.

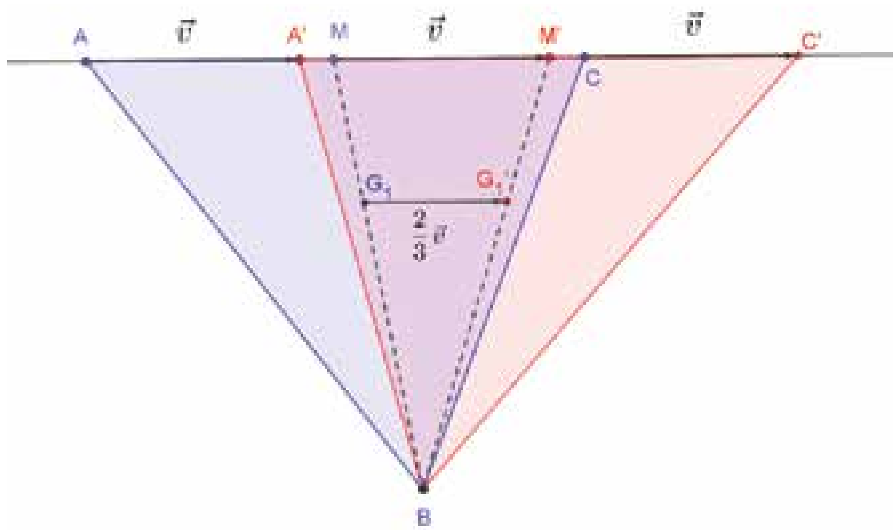


Figure 6. If $A' = A + \boldsymbol{v}$ and $C' = C + \boldsymbol{v}$, then $G_1' = G_1 + \frac{2}{3}\boldsymbol{v}$.

These two facts and the following lemma may also be formulated using idea of *shear mappings*, but this is probably not very well-known at school. Knowledge of shear mappings is not necessary; we can do without it (the *intercept theorem* or *homothety* suffice).

Lemma 2: Let $ABCD$ be a quadrilateral with interior diagonal AC. Let the points A, C be translated along AC by a vector $\boldsymbol{v}$ to A', C'; let the points B, D be translated by $-\boldsymbol{v}$ to B', D'. Then quadrilateral $A'B'C'D'$ has the same vertex centroid as quadrilateral $ABCD$, and the laminar centroid G_L of $ABCD$ maps to the laminar centroid G_L' of $A'B'C'D'$ via the translation $\frac{1}{3}\boldsymbol{v}$ (see Figure 7).

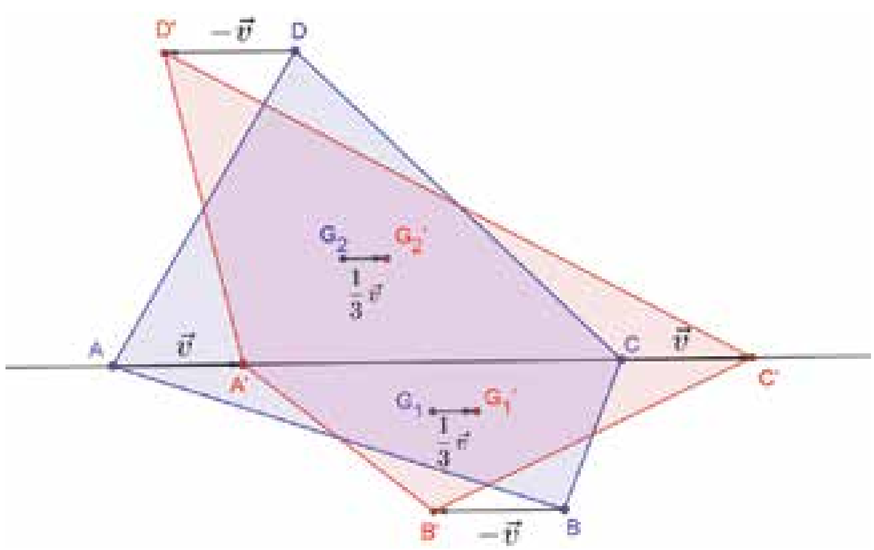


Figure 7. To Lemma 2

Proof of Lemma 2: The first claim in Lemma 2 is immediately clear because the total shift of all four points together is **0**.

From the facts **a)** and **b)** presented earlier, the shifts $G_1 \longmapsto G_1'$ and $G_2 \longmapsto G_2'$ of the centroids of the triangles are given by $\frac{1}{3}\boldsymbol{v}$. Now by Lemma 1, the laminar centroid G_L of $ABCD$ divides line segment G_1G_2 in the same ratio as the laminar centroid G_L' of $A'B'C'D'$ divides line segment $G_1'G_2'$. This ratio is given by the *areas* of the two triangles; these are the weights. But since the areas of the triangles do not change (same base and same altitude), the ratios are the same! It follows that the shift $G_{L'} \longrightarrow G_L'$ is given by $\frac{1}{3}\boldsymbol{v}$, too. ■

Now we are ready to **prove** the 'only if' part of the **Theorem**. Let $ABCD$ be a quadrilateral with interior diagonal AC and the property $G_L = G_V$. We want to prove that it must be a parallelogram. We apply the operation of Lemma 2 to $ABCD$, choosing the vector $\boldsymbol{v}$ along AC in such a way that the diagonal $A'C'$ of the new quadrilateral is *bisected* by the other diagonal $B'D'$ at points $M' = N'$ (for this, we choose $\boldsymbol{v} = \frac{1}{2}\boldsymbol{MN}$ where N denotes the intersection point of AC and BD; see Figure 8).

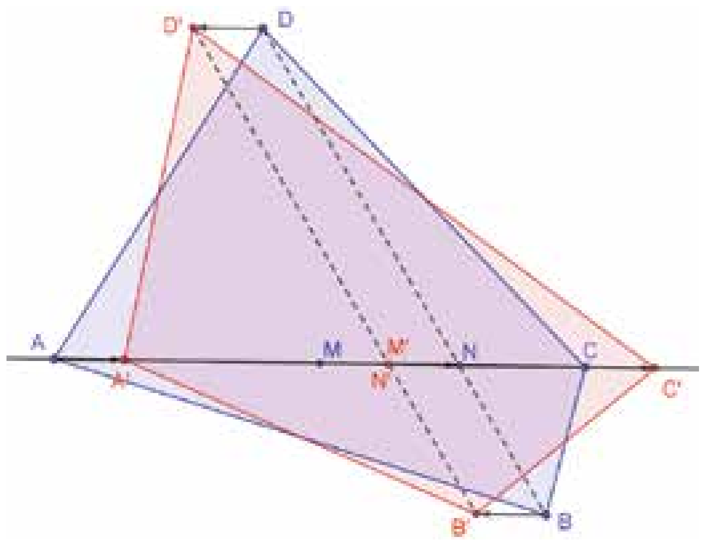


Figure 8. The operation of Lemma 2 with $\boldsymbol{v} = \frac{1}{2}\boldsymbol{MN}$.

After this operation $B'D'$ will surely be an interior diagonal of the quadrilateral $A'B'C'D'$, and again we apply the operation of Lemma 2, this second time with vector $\boldsymbol{w}$ parallel to $B'D'$, and we choose $\boldsymbol{w}$ in such a way that the diagonal $B''D''$ of the image quadrilateral $A''B''C''D''$is bisected by the other diagonal $A''C''$. Now both diagonals bisect each other, which means that $A''B''C''D''$ must be a parallelogram. We know that in parallelograms $G_L = G_V$ holds, so

$$G_L\,(A''B''C''D'') = G_V\,(A''B''C''D'')\,.$$

Since the vertex centroid did not change when applying the two operations, we know that

$$G_V\,(A''B''C''D'') = G_V\,(ABCD)\,,$$

hence $G_L\,(A''B''C''D'') = G_V\,(ABCD)\,.$

On the other hand, we know that

$$G_L\,(A''B''C''D'') = G_L\,(ABCD) + \frac{1}{3}\boldsymbol{v} + \boldsymbol{w},$$

and since $G_V\,(ABCD) = G_L(ABCD)$, this means that this added shift vector $\frac{1}{3}\boldsymbol{v} + \boldsymbol{w}$ must vanish. This vanishes only for $\boldsymbol{v} = \boldsymbol{0} = \boldsymbol{w}$, which means that $ABCD$ is a parallelogram. ■

This was, roughly spoken (we made some additional sketches and did not translate literally), the version of Kirsch 1987. Then K. Seebach (Munich) came up with another purely geometric proof (Seebach 1994) using the principle of *homothety*. And it was again A. Kirsch, 1995, who made this proof still easier and shorter, and he used the words (translated from German) "hereby probably the ideal geometric proof of the statement is found!"

This proof needs the knowledge how to construct the *laminar centroid* of a quadrilateral $ABCD$. First, draw the diagonal AC and the laminar centroids of ΔABC and ΔADC, i.e., $G_L(ABC)$ and $G_L(ADC)$. The laminar centroid $G_L(ABCD)$ of the whole quadrilateral must lie on the line segment connecting $G_L(ABC)$ and $G_L(ADC)$ (we even know where, but now that is not important).

Doing the same with the other diagonal BD, we know that $G_L(ABCD)$ must be the point of intersection of the segments $G_L(ABC)\ G_L(ADC$ and $G_L(ABD)\ G_L(CBD)$. One also must know how to construct the vertex centroid of a quadrilateral $ABCD$: it is the midpoint of the line segment joining the midpoints of the two diagonals. These two principles were already used in Figure 3.

Assume that $ABCD$ is not a parallelogram. Let S be the intersection point of the diagonals (Figure 9); then S is not simultaneously the midpoint of both the diagonals.

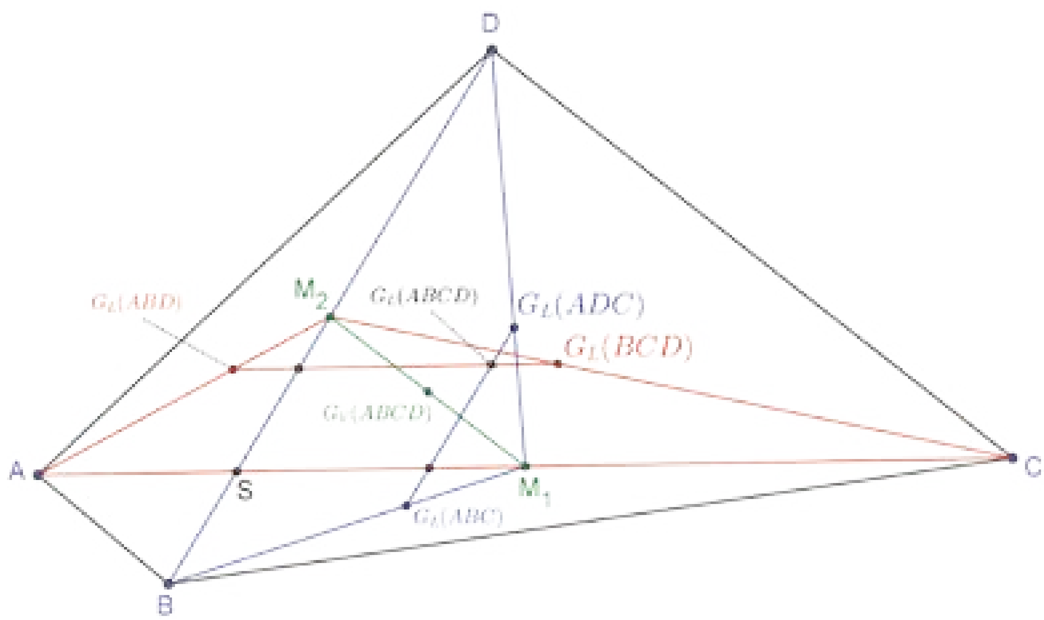


Figure 9. Very short and purely geometric proof – *convex* case

Then, using the well-known properties of triangle centroids, the intercept theorem, and its converse, one can see immediately (note the parallelogram with opposite vertices S and $G_L(ABCD)$:

$$\boldsymbol{SG_L}(ABCD) = \frac{2}{3}\boldsymbol{SM_1} + \frac{2}{3}\boldsymbol{SM_2} = \frac{2}{3}(\boldsymbol{SM_1} + \boldsymbol{SM_2}) \neq \frac{1}{2}(\boldsymbol{SM_1} + \boldsymbol{SM_2}) = \boldsymbol{SG_V}(ABCD), \qquad (*)$$

hence $G_L(ABCD) \neq G_V(ABCD)$.

Remarks

- Note that our precondition is that while we do not have $M_1 = S = M_2$(because $ABCD$ is not a parallelogram), the case $M_1 = S \neq M_2$ is covered by the above.
- The use of vector notation in (*) is just for abbreviation; one could easily avoid it and describe with more words the resulting parallelogram with opposite vertices S and $G_L(ABCD)$. Thus, this proof can be seen as purely geometric, and not analytic, although we used vectors in (*).

In the concave case nearly nothing changes (Figure 10), the only difference is that S lies in the exterior of $ABCD$ and the two triangles ΔABD and ΔCBD are not "addeD " (for getting the quadrilateral $ABCD$) but "subtracteD ".

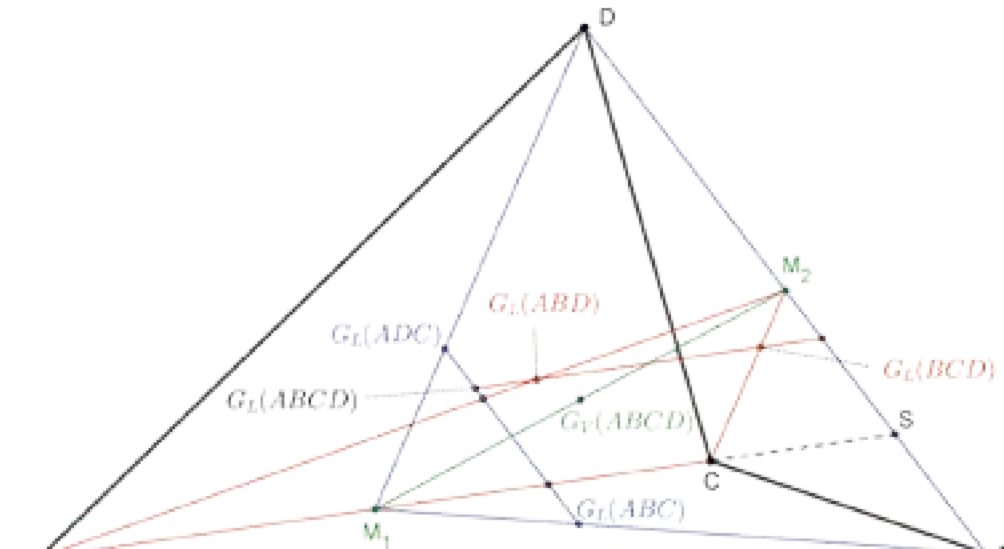


Figure 10. Short and purely geometric proof – *concave* case

Here is another short proof of the **Theorem** with *coordinates*, *vectors,* and an *oblique coordinate system*.

We use an oblique coordinate system. The origin lies in the intersection point of the diagonals of the quadrilateral. The first axis is the straight line AC and the second BD. Then the vertices are:

$A = (a, 0);$	$B = (0, b), b < 0;$	$C = (c, 0), c > a;$	$D = (0, d), d > 0.$

Then the *vertex centroid* is given by

$$G_V = \left(\frac{a+c}{4}, \frac{b+d}{4} \right).$$

The centroid of the ΔABC is

$$G_1 = \left(\frac{a+c}{3}, \frac{b}{3} \right),$$

and the centroid of ΔADC is

$$G_2 = \left(\frac{a+c}{3}, \frac{d}{3} \right).$$

According to Lemma 1, the laminar centroid G_L of the quadrilateral $ABCD$ is the *weighted mean* of the points G_1 and G_2, where the weights are the triangle areas or weights proportional to these areas, namely, $-b$ and d:

$$G_L = \frac{-b}{(-b)+d} \cdot \left(\frac{a+c}{3}, \frac{b}{3} \right) + \frac{d}{(-b)+d} \cdot \left(\frac{a+c}{3}, \frac{d}{3} \right) = \left(\frac{a+c}{3}, \frac{b+d}{3} \right).$$

Hence, $G_L = G_V$ holds if and only if $a = -c$ and $b = -d$, i.e., $ABCD$ is a parallelogram. ■

Conclusion

In many cases school students get wrong impressions concerning centroids (e.g., that there is only one kind of centroid, or that if distinguished at all, the *laminar centroid* necessarily coincides with the *vertex centroid*, as with triangles). Dealing with that topic in case of quadrilaterals (how to determine the laminar centroid of a quadrilateral, parallelograms have the property $G_L = G_V$, and only they have this property, and so on) provides a possible way to prevent this misconception. Many proofs for "only parallelograms have this property" are too complicated to be treated at school but Kirsch's proofs are elementary, purely geometric and students can easily follow every single step. Of course, it cannot be expected that students find these steps on their own; this was an ingenious idea of Arnold Kirsch. And using the alternative proof using *analytic geometry* provides a good opportunity to make use of oblique coordinate systems.

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HANS HUMENBERGER received his PhD at the University of Vienna and later his habilitation in the field of *mathematics education*. He works at his alma mater as a professor for mathematics with special emphasis on mathematics education. He also heads the working group *didactics of mathematics and school mathematics* and he, along with his team, is responsible for the education of teachers at secondary and high school level. He has written many papers in German and English in mathematics education and mathematics, and also several books, most of them in German. His main fields of interest are mathematics as a process, applications of mathematics, problem solving, geometry and stochastics. For more details see his homepage: https://homepage.univie.ac.at/hans.humenberger/ Email: hans.humenberger@univie.ac.at University of Vienna, Faculty for Mathematics, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria

Geometric and Calculus Proofs of Some Inequalities

SARMAD HOSSAIN

Introduction

There are many visual and calculus-based proofs of $\pi^e < e^\pi$ and $b^a < a^b$(for $e \le a < b$). The aim of this paper is to give geometric and calculus-based proofs of these inequalities. We show in addition that $b^a > a^b$ for $0 < a < b \le e$ and $a^b \le 1 < b^a$ for $0 < a \le 1 < b$.

To show $\pi^e < e^\pi$, Nelson ([2]) uses the fact that the curve $y = e^{x/e}$ lies above the line $y = x$, while Nakhli ([1]) uses the fact that the curve $y = \frac{\ln x}{x}$ attains the global maximum at the point e.

Proofs using the curve $y = \frac{1}{x}$

Here we use the curve $y = \frac{1}{x}$ and the fact that the shaded region in Figure 1 lies within the rectangle bounded by the lines $x = \frac{1}{\pi}$, $x = \frac{1}{e}$, the x-axis and the line $y = \pi$.

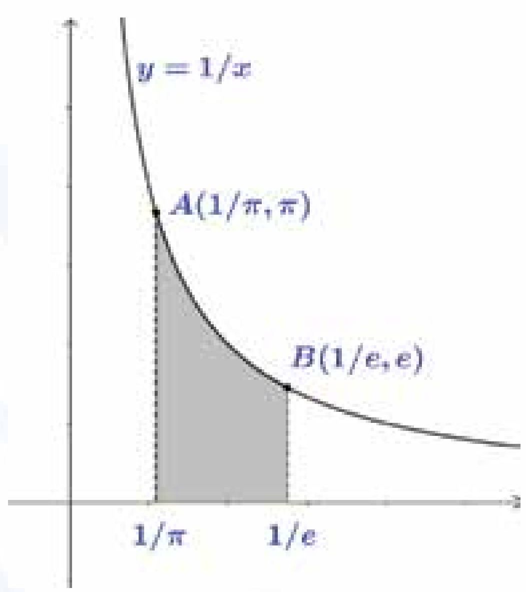

Figure 1. Geometric visualisation of $\pi^e < e^\pi$

Keywords: Proofs, graphs, calculus, inequalities.

From Figure 1, as $e < \pi \Rightarrow \frac{1}{\pi} < \frac{1}{e}$, we have

$$\ln\left(\frac{1}{e}\right) - \ln\left(\frac{1}{\pi}\right) = \int_{\frac{1}{\pi}}^{\frac{1}{e}} \frac{1}{x}dx < \pi\left(\frac{1}{e} - \frac{1}{\pi}\right),$$

$$\Rightarrow -\ln e + \ln \pi < \frac{\pi - e}{e} \Rightarrow \ln \pi - 1 < \frac{\pi}{e} - 1 \Rightarrow \ln \pi < \frac{\pi}{e} \Rightarrow \pi^e < e^\pi$$

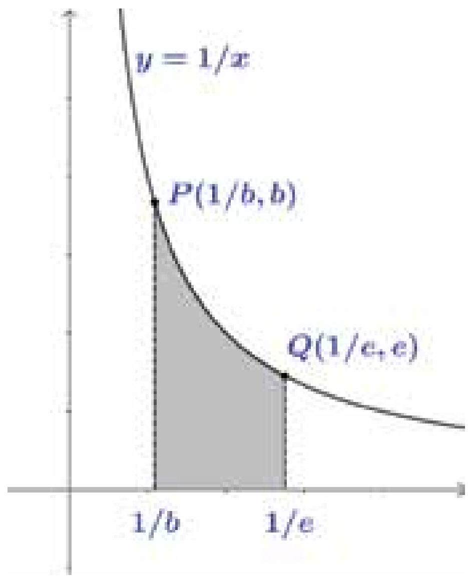

Figure 2. Geometric visualization of $b^e < e^b$ for $e < b$

From Figure 2, as $e < b \Rightarrow \frac{1}{b} < \frac{1}{e}$, we have

$$\ln\left(\frac{1}{e}\right) - \ln\left(\frac{1}{b}\right) = \int_{1/b}^{1/e} \frac{1}{x}dx < b\left(\frac{1}{e} - \frac{1}{b}\right)$$

$$\Rightarrow -\ln \mathrm{e} + \ln b < \frac{b - e}{e} \Rightarrow \ln b - 1 < \frac{b}{e} - 1 \Rightarrow \ln b < \frac{b}{e} \Rightarrow b^e < e^b$$

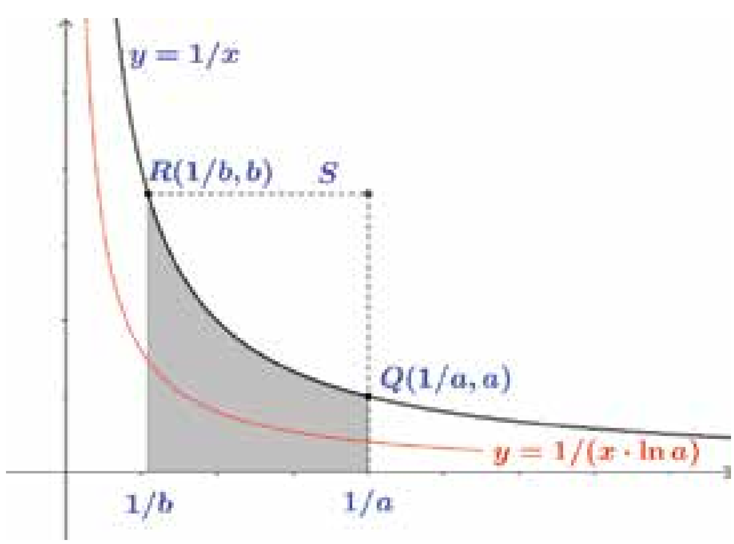

Figure 3. Geometric visualization of $b^a < a^b$ for $e \leq a < b$

From Figure 3, as $e \le a < b \Rightarrow \frac{1}{b} < \frac{1}{a} \le \frac{1}{e}$, we have

$$\frac{\ln(1/a)}{\ln a} - \frac{\ln(1/b)}{\ln a} = \int_{1/b}^{1/a} \left(\frac{1}{x \ln a}\right) dx < b\left(\frac{1}{a} - \frac{1}{b}\right),$$
$$\Rightarrow \frac{-\ln a}{\ln a} + \frac{\ln b}{\ln a} < \frac{b-a}{a} \Rightarrow \frac{\ln b}{\ln a} - 1 < \frac{b}{a} - 1$$
$$\Rightarrow \frac{\ln b}{\ln a} < \frac{b}{a} \Rightarrow a \ln b < b \ln a \Rightarrow b^a < a^b$$

Proof using calculus

Consider the function $f(x) = \left(\frac{1}{x}\right)^x$. Taking natural logarithms on both sides we get,

$\ln f(x) = x \ln\left(\frac{1}{x}\right) = -x \cdot \ln x$

$\Rightarrow \frac{f'(x)}{f(x)} = -\ln x - 1 = -(1 + \ln x)$

$\Rightarrow f'(x) = -f(x)\,(1 + \ln x)$.

Now, if $\ln x + 1 < 0 \Rightarrow \ln x < -1 \Rightarrow x < e^{-1} \Rightarrow x < \frac{1}{e}$.

Similarly, if $\ln x + 1 > 0 \Rightarrow x > \frac{1}{e}$ and $\ln x + 1 = 0 \Rightarrow x = \frac{1}{e}$.

Therefore, $x \in \left(0, \frac{1}{e}\right] \Rightarrow f'(x) \ge 0 \Rightarrow f(x)$ is strictly increasing in $\left(0, \frac{1}{e}\right)$

Similarly, $x \in \left[\frac{1}{e}, \infty\right) \Rightarrow f'(x) \le 0 \Rightarrow f(x)$ is strictly decreasing in $\left(\frac{1}{e}, \infty\right)$.

Case 1. $e \le a < b$

Now, $e \le a < b \Rightarrow \frac{1}{b} < \frac{1}{a} \le \frac{1}{e} \Rightarrow \left(\frac{1}{1/b}\right)^{1/b} < \left(\frac{1}{1/a}\right)^{1/a}$

$\left(\text{since } f(x) \text{ is strictly increasing in } \left(0, \frac{1}{e}\right)\right)$

$\Rightarrow b^{1/b} < a^{1/a} \Rightarrow b^a < a^b$.

Case 2. $0 < a < b \le e$

Now, $0 < a < b \le e \Rightarrow \frac{1}{e} \le \frac{1}{b} < \frac{1}{a} < \infty \Rightarrow \left(\frac{1}{1/b}\right)^{1/b} > \left(\frac{1}{1/a}\right)^{1/a}$

$\left(\text{since } f(x) \text{ is strictly decreasing in } \left(\frac{1}{e}, \infty\right)\right)$

$\Rightarrow b^{1/b} > a^{1/a} \Rightarrow b^a > a^b$.

Case 3. $0 < a \le 1 < b$

Now, $0 < a \le 1 < b \Rightarrow \frac{1}{b} < 1 \le \frac{1}{a} < \infty \Rightarrow \ln\left(\frac{1}{b}\right) < \ln 1 \le \ln\left(\frac{1}{a}\right) \Rightarrow \ln\left(\frac{1}{b}\right) < 0 \le \ln\left(\frac{1}{a}\right)$

$\Rightarrow -\ln b < 0$ and $0 \le -\ln a \Rightarrow \ln b > 0$ and $0 \ge \ln a$

$\Rightarrow a \ln b > 0$, and $0 \ge b \ln a$

$\Rightarrow b^a > 1$ and $a^b \le 1 \Rightarrow a^b \le 1 < b^a$.

Remark 1. $\pi^e < e^\pi$

As $e < \pi$, by Case 1, we have $\pi^e < e^\pi$.

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SARMAD HOSSAIN is a Research Scholar at the Department of Mathematics, Ramakrishna Mission Vivekananda Centenary College, Rahara, Kolkata. He may be contacted at sarmad786hossain@gmail.com

Maxima and Minima using the Power Mean Inequality

TOYESH PRAKASH SHARMA &
ETISHA SHARMA

The power mean inequality states the following:

For positive quantities $a_1, a_2, \cdots, a_n$ *and* $p \geq 1$,

$$\frac{a_1{}^p + a_2{}^p + \cdots + a_n{}^p}{n} \geq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^p.$$

For $0 \leq p \leq 1$, *inequality reverses.*

Proof

We know that for $p \geq 1, f(x) = x^p$ is convex; the graph has the following appearance (Figure 1):

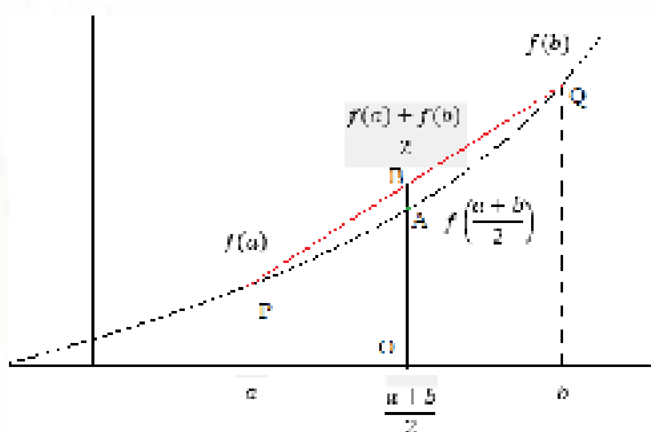

Figure 1

From Figure 1 we have

$$OB \geq OA,$$

$$\Rightarrow \frac{f(a) + f(b)}{2} \geq f\left(\frac{a+b}{2}\right).$$

Here $f(a) = a^p, f(b) = b^p$ and $f\left(\frac{a+b}{2}\right) = \left(\frac{a+b}{2}\right)^p$. Hence

$$\frac{a^p + b^p}{2} \geq \left(\frac{a+b}{2}\right)^p.$$

This is the power mean inequality for two variables. Now let $a = \frac{a_1 + a_2}{2}$ and $b = \frac{b_1 + b_2}{2}$. Then

$$\frac{\left(\frac{a_1+a_2}{2}\right)^p + \left(\frac{b_1+b_2}{2}\right)^p}{2} \geq \left(\frac{\frac{a_1+a_2}{2} + \frac{b_1+b_2}{2}}{2}\right)^p.$$

Using the power mean inequality for $\left(\frac{a_1+a_2}{2}\right)^p$ and $\left(\frac{b_1+b_2}{2}\right)^p$ we get:

$$\frac{\frac{{a_1}^p + {a_2}^p}{2} + \frac{{b_1}^p + {b_2}^p}{2}}{2} \geq \frac{\left(\frac{a_1+a_2}{2}\right)^p + \left(\frac{b_1+b_2}{2}\right)^p}{2}.$$

So,

$$\frac{\frac{{a_1}^p + {a_2}^p}{2} + \frac{{b_1}^p + {b_2}^p}{2}}{2} \geq \left(\frac{\frac{a_1+a_2}{2} + \frac{b_1+b_2}{2}}{2}\right)^p,$$

$$\Rightarrow \frac{{a_1}^p + {a_2}^p + {b_1}^p + {b_2}^p}{4} \geq \left(\frac{a_1 + a_2 + b_1 + b_2}{4}\right)^p.$$

This is power mean inequality for four variables. Similarly for n quantities we have:

$$\frac{{a_1}^p + {a_2}^p + \cdots + {a_n}^p}{n} \geq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^p; p \geq 1.$$

For $0 \leq p \leq 1$ the same function i.e., $f(x) = x^p$ is concave function, so $\frac{f(a)+f(b)}{2} \leq f\left(\frac{a+b}{2}\right)$ and

$$\frac{{a_1}^p + {a_2}^p + \cdots + {a_n}^p}{n} \geq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^p; 0 \leq p \leq 1.$$

Problem 1. Find the maximum and minimum values of the function $\sin x + \cos x$.

Solution

The power mean inequality states that for real positive quantities a, b and $p \geq 1$,

$$\frac{a^p + b^p}{2} \geq \left(\frac{a+b}{2}\right)^p.$$

Letting $a = \sin x$ and $b = \cos x$ and $p = 2$, we get

$$\frac{\sin^2 x + \cos^2 x}{2} \geq \left(\frac{\sin x + \cos x}{2}\right)^2 \Rightarrow \frac{1}{2} \geq \left(\frac{\sin x + \cos x}{2}\right)^2$$

$$\Rightarrow 2 \geq (\sin x + \cos x)^2 = \begin{cases} \sin x + \cos x \geq -\sqrt{2} \\ \sin x + \cos x \leq \sqrt{2} \end{cases}$$

$\therefore \sqrt{2}$ is the maximum value of $\sin x + \cos x$ and $-\sqrt{2}$ is the minimum value of $\sin x + \cos x$.

The above problem is given as an exercise in a chapter of class 12 entitled as "Application of Derivatives" [1].

Problem 2. For $a, n \geq 1$, find the maximum value of $\sqrt[n]{\sin ax} + \sqrt[n]{\cos ax}$.

(Note: This is meant to be done without using calculus.)

Solution

The power mean inequality states for real quantities u, v and $p \geq 1$,

$$\frac{u^{1/p} + v^{1/p}}{2} \leq \left(\frac{u+v}{2}\right)^{1/p}.$$

Letting $u = \sin^2 ax$, $v = \cos^2 ax$ and $p = 2n$ gives

$$\frac{\left(\sin^2 ax\right)^{1/2n} + \left(\cos^2 ax\right)^{1/2n}}{2} \leq \left(\frac{\sin^2 ax + \cos^2 ax}{2}\right)^{1/2n},$$

$$\text{hence: } \sqrt[n]{\sin ax} + \sqrt[n]{\cos ax} \leq 2\left(\frac{1}{2}\right)^{\frac{1}{2n}} = 2^{1-1/2n}.$$

Hence the maximum value of the function is $2^{1-1/2n}$.

The next problem was proposed by Jose L.D-Barrero [2]

Problem 3. Let a, b, c, d be four positive real numbers. Find the minimum value of

$$\frac{\sqrt[4]{a} + \sqrt[4]{b} + \sqrt[4]{c} + \sqrt[4]{d}}{\sqrt[4]{a+b+c+d}}.$$

Solution

From the power mean inequality:

$$\frac{\sqrt[4]{a} + \sqrt[4]{b} + \sqrt[4]{c} + \sqrt[4]{d}}{4} \leq \left(\frac{a+b+c+d}{4}\right)^{\frac{1}{4}},$$

$$\Rightarrow \frac{\sqrt[4]{a} + \sqrt[4]{b} + \sqrt[4]{c} + \sqrt[4]{d}}{\sqrt[4]{a+b+c+d}} \leq 4\left(\frac{1}{4}\right)^{\frac{1}{4}} = 2\sqrt{2}$$

Thus, the maximum value of $\frac{\sqrt[4]{a} + \sqrt[4]{b} + \sqrt[4]{c} + \sqrt[4]{d}}{\sqrt[4]{a+b+c+d}}$ is $2\sqrt{2}$ which is attained when $a = b = c = d$.

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TOYESH PRAKASH SHARMA has been interested in science, mathematics, and literature since high school. He has contributed mathematics articles to magazines such as *Mathematical Gazette*, *Crux Mathematicorum*, *Parabola*, *AMJ*, *ISROSET*, *SSMJ*, *Pentagon*, *Octagon*, *La Gaceta de la RSME*, *At Right Angles*, *Fibonacci Quarterly*, *Mathematical Reflections*, *Irish Mathematical Society*, *Indian Mathematical Society*, and *Mathematical Student*. He has also written two books for high school students, "Problems on Trigonometry" and "Problems on Surds." Currently he is doing his B Sc in Physics and Mathematics from Agra College, Agra, India. He may be contacted at toyeshprakash@gmail.com.

ETISHA SHARMA is interested in Mathematics, Computer and Drawing. She completed her schooling from Gayatri Public School, Agra. Currently she is doing her B Sc in Mathematics, Physics and Chemistry from Agra College, Agra, India. She may be contacted at etisha20020830gmail.com.

Problems based on the AM-GM Inequality - Part II

TOYESH PRAKASH SHARMA

In this two-part article, we consider problems from various sources which are solved using the AM-GM inequality. We list the problems first and give the solutions later. We continue from where we left off in Part I.

Problems

Problem 4. Given that x, y, z are positive real numbers and $xy + yz + zx = 1$, find the least value of

$$\frac{x^3}{x^2+y^2} + \frac{y^3}{y^2+z^2} + \frac{z^3}{z^2+x^2}. \tag{4}$$

This problem is from the *SSMJ* Problem corner, December 2014; it was proposed by Arkady Alt [5].

Problem 5. Let a, b, c be positive real numbers. Prove that

$$\sqrt{a^2+ca} + \sqrt{b^2+bc} + \sqrt{c^2+ca} \le \sqrt{2}(a+b+c). \tag{5}$$

This problem was published in *Crux Mathematicorum*; it was proposed by Jose Luiz Diaz Barrero [6].

Keywords: AM-GM inequality

Problem 6. Let x, y, z be positive real numbers such that $x+y+z=3$. Prove that

$$\frac{x^4+x^2+1}{x^2+x+1}+\frac{y^4+y^2+1}{y^2+y+1}+\frac{z^4+z^2+1}{z^2+z+1} \geq 3xyz. \tag{6}$$

I found this problem at https://www.mat.uniroma2.it/ tauraso/AMM/AMM11815.pdf; it was proposed by G. Apostolopoulos in *American Mathematical Monthly*, [7].

Problem 7. Let a, b, c be three positive real numbers such that $ab+bc+ca=2abc$. Prove that

$$\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}} \leq 2. \tag{7}$$

Solutions

Problem 4. Given that x, y, z are positive real numbers and $xy+yz+zx=1$, find the least value of

$$\frac{x^3}{x^2+y^2}+\frac{y^3}{y^2+z^2}+\frac{z^3}{z^2+x^2}.$$

Solution. From the AM-GM inequality, we have

$$\frac{x^3}{x^2+y^2}=x-\frac{xy^2}{x^2+y^2} \geq x-\frac{xy^2}{2xy}=x-\frac{y}{2}.$$

Similarly, we have

$$\frac{y^3}{y^2+z^2} \geq y-\frac{z}{2},$$
$$\frac{z^3}{z^2+x^2} \geq z-\frac{x}{2}.$$

Adding these three inequalities, we get

$$\begin{aligned}\frac{x^3}{x^2+y^2}+\frac{y^3}{y^2+z^2}+\frac{z^3}{z^2+x^2} &\geq \frac{x+y+z}{2}=\frac{3}{2}\cdot\frac{x+y+z}{3}\\ &\geq \frac{3}{2}\cdot\left(\frac{xy+yz+zx}{3}\right)^{1/2}=\frac{\sqrt{3}}{2}.\end{aligned}$$

Hence the minimum value of $\frac{x^3}{x^2+y^2}+\frac{y^3}{y^2+z^2}+\frac{z^3}{z^2+x^2}$ is $\frac{1}{2}\sqrt{3}$.

Problem 5. Let a, b, c be positive real numbers. Prove that

$$\sqrt{a^2+ca}+\sqrt{b^2+bc}+\sqrt{c^2+ca} \leq \sqrt{2}(a+b+c).$$

Solution. We have:

$$\begin{aligned}\sqrt{a^2+ca}+\sqrt{b^2+bc}+\sqrt{c^2+ca} &= \frac{1}{\sqrt{2}}\cdot\left(\sqrt{2a\cdot(a+c)}+\sqrt{2b\cdot(b+c)}+\sqrt{2c\cdot(c+a)}\right)\\ &\le \frac{1}{\sqrt{2}}\cdot\left(\frac{2a+(a+c)}{2}+\frac{2b+(b+c)}{2}+\frac{2c+(c+a)}{2}\right)\\ &= \frac{1}{2\cdot\sqrt{2}}\cdot(4a+4b+4c)=\sqrt{2}\cdot(a+b+c).\end{aligned}$$

Problem 6. Let x, y, z be positive real numbers such that $x+y+z=3$. Prove that

$$\frac{x^4+x^2+1}{x^2+x+1}+\frac{y^4+y^2+1}{y^2+y+1}+\frac{z^4+z^2+1}{z^2+z+1}\ge 3xyz.$$

Solution. We have

$$\begin{aligned}\frac{x^4+x^2+1}{x^2+x+1}+\frac{y^4+y^2+1}{y^2+y+1}+\frac{z^4+z^2+1}{z^2+z+1} &= \left(x^2-x+1\right)+\left(y^2-y+1\right)+\left(z^2-z+1\right)\\ &= 3\cdot\frac{x^2+y^2+z^2}{3}\ge 3\cdot\left(\frac{x+y+z}{3}\right)^2=3.\end{aligned}$$

Also:

$$3=3\cdot\left(\frac{x+y+z}{3}\right)^3\ge 3xyz,$$

hence proved.

Problem 7. Let a, b, c be three positive real numbers such that $ab+bc+ca=2abc$. Prove that

$$\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}\le 2.$$

Solution (published in 2019 in the issue 1 of AMJ journal). From $ab+bc+ca=2abc$ we get

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=2.$$

Next, using the AM-GM inequality, we get

$$\begin{aligned}\frac{1}{a}+\frac{1}{b} &\ge \frac{2}{\sqrt{ab}},\\ \frac{1}{b}+\frac{1}{c} &\ge \frac{2}{\sqrt{bc}},\\ \frac{1}{c}+\frac{1}{a} &\ge \frac{2}{\sqrt{ca}}.\end{aligned}$$

By addition we get

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}},$$

and the stated result follows.

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TOYESH PRAKASH SHARMA has been interested in science, mathematics, and literature since high school. He has contributed mathematics articles to magazines such as *Mathematical Gazette*, *Crux Mathematicorum*, *Parabola*, *AMJ*, *ISROSET*, *SSMJ*, *Pentagon*, *Octagon*, *La Gaceta de la RSME*, *At Right Angles*, *Fibonacci Quarterly*, *Mathematical Reflections*, *Irish Mathematical Society*, *Indian Mathematical Society*, and *Mathematical Student*. He has also written two books for high school students, "Problems on Trigonometry" and "Problems on Surds." Currently he is doing his B Sc in Physics and Mathematics from Agra College, Agra, India. He may be contacted at toyeshprakash@gmail.com.

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Prospective authors are asked to observe the following guidelines.

1. Use a readable and inviting style of writing which attempts to capture the reader's attention at the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, figure with an interesting question or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps which depend on hidden calculations.
5. Avoid specialized jargon and notation — terms that will be familiar only to specialists. If technical terms are needed, please define them.
6. Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. Provide a compact list of references, with short recommendations.
8. Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. Number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note, the minimum resolution for photos or scanned images should be 300dpi).
12. Refer to diagrams, photos, and figures by their numbers and avoid using references like 'here' or 'there' or 'above' or 'below'.
13. Include a high resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. Submit articles in MS Word format or in LaTeX.

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A Call for Articles

Classroom teachers are at the forefront of helping students grasp core topics. Students with a strong foundation are better able to use key concepts to solve problems, apply more nuanced methods, and build a structure that help them learn more advanced topics.

The focal theme of this section of At Right Angles (AtRiA) is the teaching of various foundational topics in the school mathematics curriculum. In relation to these topics, it addresses issues such as knowledge demands for teaching, students' ideas as they come up in the classroom and how to build a connected understanding of the mathematical content.

Foundational topics include, but are not limited to, the following:

- Number systems, patterns and operations
- Fractions, ratios and decimals
- Proportional reasoning
- Integers
- Bridging Arithmetic-Algebra
- Geometry
- Measurement and Mensuration
- Data Handling
- Probability

We invite articles from teachers, teacher educators and others that are helpful in designing and implementing effective instruction. We strongly encourage submissions that draw directly on experiences of teaching. This is an opportunity to share your successful teaching episodes with AtRiA readers, and to reflect on what might have made them successful. We are also looking for articles that strengthen and support the teachers' own understanding of these topics and strengthen their pedagogical content knowledge.

Articles in this section may address key questions such as -

- What challenges did your students face while learning these fundamental mathematical topics?
- What approaches that you used were successful?
- What preparations, in terms of knowing mathematics, enacting the tasks and analysing students work were needed for effective instruction?
- What contexts, representations, models did you use that facilitated meaning making by your students?

Send in your articles to
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The magazine has zero tolerance for plagiarism. By submitting an article for publishing, the author is assumed to declare it to be original and not under any legal restriction for publication (e.g. previous copyright ownership). Wherever appropriate, relevant references and sources will be clearly indicated in the article.

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NEWSPAPERS IN THE MATHEMATICS CLASS

PADMAPRIYA SHIRALI

Azim Premji University

A publication of Azim Premji University
together with Community Mathematics Centre,
Rishi Valley

NEWSPAPERS IN THE MATHEMATICS CLASS

By the time students reach middle school, they are about twelve years old or more and have acquired basic literacy and numeracy skills. They can comprehend simple text and process information presented in tabular form or graphs. Their awareness of the world — geographically and culturally — is starting to grow. They begin to ask questions about the world around them.

Newspapers can serve as a good resource for middle school students to do real mathematics. For a major part of the time, students solve problems from their textbooks. While these problems may be realistic and drawn from life situations, they are not real time problems. On the other hand, newspapers have current data and report on the latest issues. The newness of it and the timeliness of it has a certain charm and a teacher's task to introduce problems based on the current contexts becomes easier as the students would find these to be more interesting and topical.

This data can be drawn from political, scientific, financial, or social areas. Using numerical data gives students a chance to glean real world mathematical information, often of subjects that are interdisciplinary in nature. They can also raise questions which bring in other subjects such as economics, history, geography, sociological issues, environmental issues, politics, etc. Students can bring to use various concepts covered in their curriculum. Financial literacy is now part of the curriculum as per the NEP, and usage of real time data from the news will make it meaningful for students.

Usage of news items may also expose students to concepts which are not part of their immediate curriculum. It is a way of learning new ideas which in turn might trigger further questions that lead to exploration of the topic.

The problems generated by discussion amongst teachers and students can break the monotony of standardized problems and generally lend themselves to strategic and lateral thinking.

While it is not possible to always work with real data, we must recognise the fact that students of the middle school

Figure 1

feel that the mathematics they do is disconnected from real life. Hence, periodically it is good to work towards helping students see the relevance of mathematics in everyday life. This will help students see more ways in which mathematics connects with their daily life.

Teachers may adopt different approaches in bringing and using items from newspapers in the classes. If the text is difficult to comprehend, teachers may like to summarise or make a simplified oral presentation of the news presented in the paper. However, to the extent possible, it is good to encourage students to read the published piece and comprehend the mathematical content involved.

Teachers can select appropriate news items over a week which will be of interest for students. They could encourage students to maintain a file for clipped items with embedded mathematical information which can be utilised for problem solving. The items selected can be mapped on to various chapters/concepts or they may be an exposure to a new aspect of mathematics for the students.

To understand such information effectively, students may need related facts which the teacher can provide by referring to the internet.

Big sports events like a Football World Cup, the ICC World Cup or the Olympics can provide a lot of math opportunities. The same applies to elections. As these are repetitive events, the ideas and approach can be used year after year.

The year's course work can end with doing a newspaper-based project which encompasses many concepts that have been covered over the year. The problems may be of an open-ended nature at times or may have multiple ways of solving them.

Keywords: Newspapers, current events, real world mathematics, data, analysis

NEWS ITEM 1 (SCIENCE NEWS)

Mathematical concepts: Ellipse, Speed

While this is a rare event, a lot of other news items about rockets and satellites connected to space exploration which appear often can be used to talk about speed, trajectory, rocket power, etc.

> Chandrayaan-3 successfully inserted into lunar orbit, says ISRO.
>
> Almost a month **since its launch**, India's third moon mission, Chandrayaan-3, was successfully inserted into the lunar orbit on Saturday, the ISRO said.
>
> During the 42 days period, the LVM3 rocket will carry its 3895-kg payload using three different rocket power stages with a maximum thrust of 10.242 km/sec (speed over 36000 km/hr) being provided by the indigenous cryogenic C-25 engine fired on the rocket in the final phase.
>
> The legs of Chandrayaan-3 have been strengthened to ensure that it would be able to land, and stabilise, even at a speed of **3 m/sec, or 10.8 km/hour**.

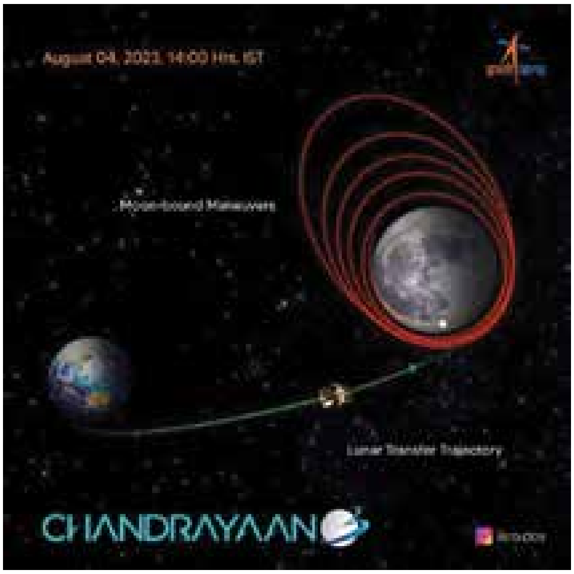


Figure 2

Here is a great opportunity to talk about orbits and the shape of orbit, an ellipse or oval.

The teacher can point out how the orbits of planets are nearly circular.

What about the orbits of satellites? Comets?

Class can try to figure out how to draw such shapes with the teacher's assistance.

To understand the speed of a rocket (36,000 km/hr) students can contrast it with the speed of a typical air flight which is 900 Km/hr. Other interesting data comparisons can also be made. Satellites generally travel at 28000 km/hr, whereas a Bullet train goes at 320 km/hr.

Related concepts of physics like thrust, basic idea of a rocket (students are familiar with Diwali rockets) and what propels it forward can be discussed. An idea of the gravity of Earth and moon will also come up in the discussion. Ideas of combustion can be talked about.

Another question that may arise is why the rocket makes three to four revolutions before entering lunar orbit.

Similar news items will make interesting read.

After Moon landing, ISRO launches its first mission to study Sun

Figure 3

NEWS ITEM 2 (NATURAL CALAMITIES)

Mathematical concept: Measurement on Richter scale, Relationship of the numbers on the scale.

Earthquake in Delhi today: Tremors felt in Delhi-NCR, other parts of North India as quake of 5.8 magnitude jolts Afghanistan

The teacher can explain what the number 5.8 refers to by explaining the basic idea of a Richter Scale.

Scientists use the numbers from 1 to 10 to say how strong an earthquake is. This number system is called a Richter scale. On this scale, 1 means a very mild earthquake which will not be experienced by anyone, but a 10 means a very high intensity earthquake. Earthquakes with a measurement of less than three are known as 'micro-quakes' and can happen multiple times a day without anyone experiencing them. The most powerful quake ever recorded was a 9.8.

It is not necessary to explain to the students about the Richter scale being a logarithmic scale as the students are not yet familiar with Logarithms. However, they can appreciate the fact that each number on the scale means a 10-fold increase. Ex. an earthquake rated as 5 is ten times as powerful as one rated as 4.

An earthquake's Richter magnitude does NOT change with distance from its source. An earthquake of, say magnitude 5 is a magnitude 5 earthquake no matter where on the globe it occurs. The ground effects, known as earthquake "intensity," die away with distance.

Students may raise further questions on what causes earthquakes and concepts involving plate tectonics and physics related to forces can be discussed.

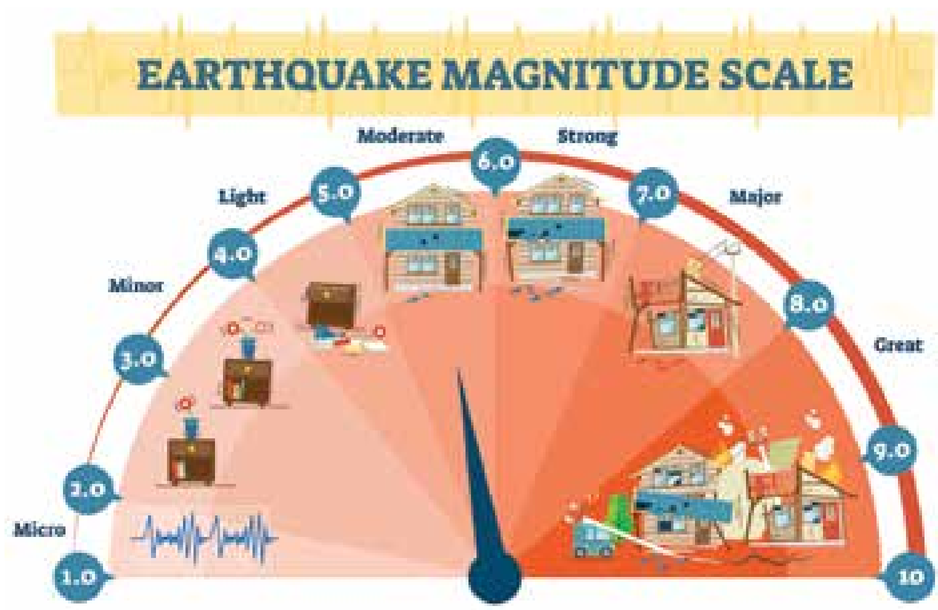


Figure 4

NEWS ITEM 3 (LARGE NUMERICAL DATA)

Mathematical concept: Large numbers, conversions from one system to another, Fractions.

'India's population in 2023 stands at 142.86 crore, according to the latest United Nations Population Fund data'.

How does one visualise such a number? Most of us live in cities or have visited a large city. We experience the jostling and the crowded feeling of a city. A typical Indian city's population is 2 to 3 crores.

Does that information help in imagining the number 142 crores? How many big cities would that be?

Teacher can extend the discussion further by bringing in more related information.

The world population is around 8 billion.

How do we compare these two figures?

Students can rewrite 142 crores in the international system. 142 crores is 142,00,00,000. When rewritten in the international system it is 1,420,000,000 that is 1 billion, 420 million.

What fraction of the world's population is in India?

Students may begin to wonder about other countries and teachers can share data of world population distribution to make a comparative study.

If information about the size of the countries is given, students will be able to figure out the density of the population, problems arising out of overcrowding, etc.

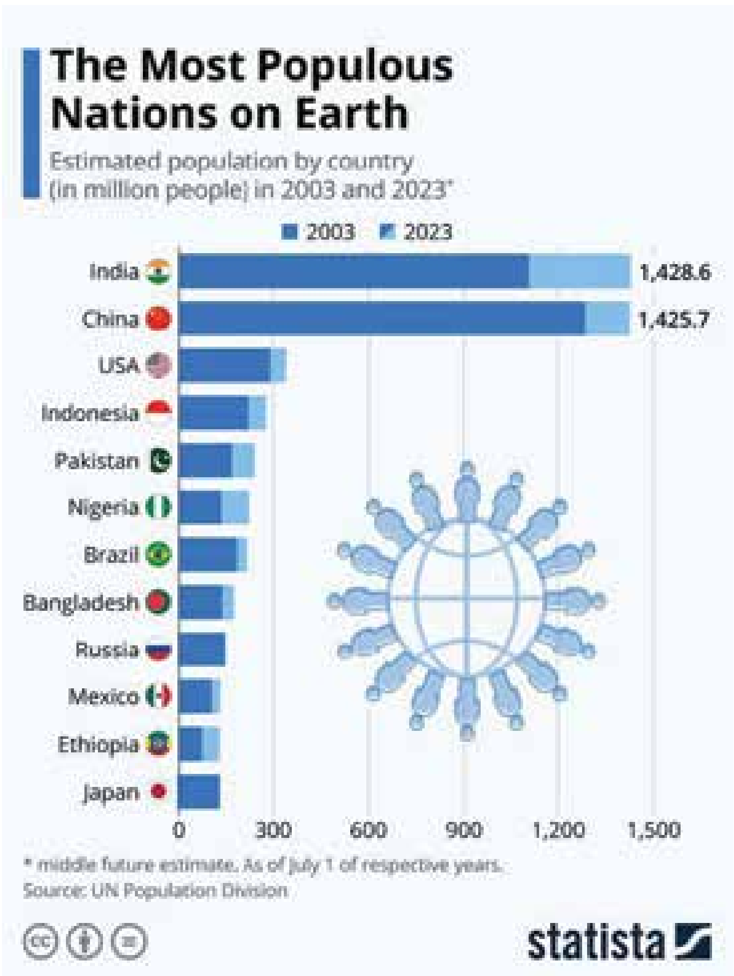


Figure 5

NEWS ITEM 4 (TEMPERATURE CHART)

Mathematical concepts: Measurement of temperature, average

Let students observe the various data items related to the weather that they notice in the weather charts.

Here is a weather chart of Pune city. What are the various data items that are given in the chart. Will the data vary from day to day?

Discuss the concept of temperature, how it is measured (centigrade). Also, what relative humidity denotes.

Why does the time of sun rise or moon rise differ over time?

Sunrise and sunset time bring in the concept of season, tilt of the earth's axis and the movement of Earth.

A lot of science can be discussed along with mathematical aspects.

Students can note down the maximum and minimum temperatures for a week and calculate the average high temperature and the average low temperature for the week. They can also represent this information as line graphs.

They can make predictions about rain or temperature for the week to follow and verify whether their predictions are close to the actual.


WEATHER

33.8°C/ 21.5°C | 33.8°C/ 21.8°C

Sunrise: 0620 | Sunset: 1850
Moonrise: 2000 | Moonset: 0726
Forecast: Partly cloudy sky, light rain likely. Max and min temperatures likely to be around 34°C & 22°C respectively
Relative humidity: Max 84% Min 62%

Figure 6

NEWS ITEM 5 (GRAPHS)

Mathematical concept: Graph interpretation, deconstructing graph, effectiveness of a graph in communication.

One finds many interesting graphs displayed along with articles in the papers. Graph reading and interpretation, studying the correspondence between the content of the article and the graph are various studies that can be taken up by students. Many questions requiring analysis can be asked of such graphs.

Example: Here is a graph depicting the GDP growth in four quarters over 3 years.

Teacher should first explain briefly to the students about GDP.

How does the growth of the first quarter of 21-22 compare with the growth of the first quarter of 22-23?

How does it compare with the growth of the last quarter of 22-23?

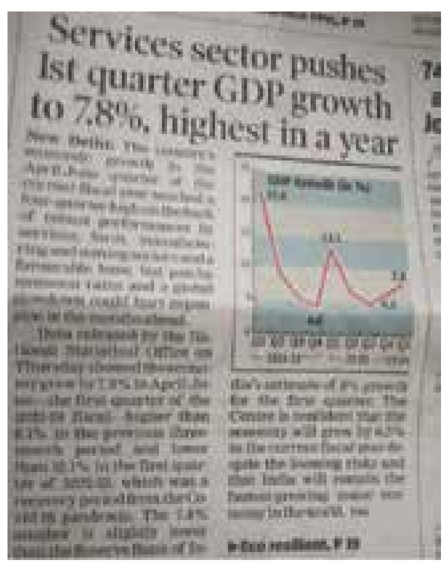

Services sector pushes Ist quarter GDP growth to 7.8%, highest in a year

Figure 7

Here is a bar graph from a newspaper, which shows the rural and urban percentage of population in poverty.

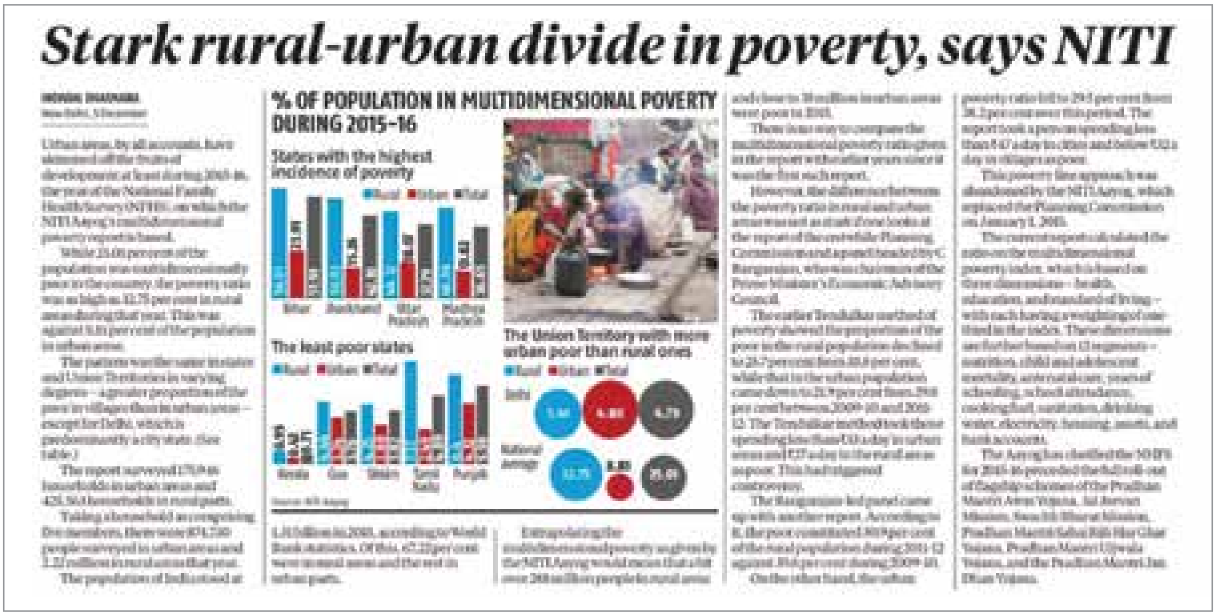
What does this graph reveal about what is happening in India?

What could be the reasons for the higher rate in the rural areas?

Do you see evidence of this in the areas that you live in?

What can we do about it?

Do you think that the graph communicates effectively the status of poverty across states? Could it have been done in any other more effective way?



Stark rural-urban divide in poverty, says NITI

% OF POPULATION IN MULTIDIMENSIONAL POVERTY DURING 2015-16

States with the highest incidence of poverty

The least poor states

The Union Territory with more urban poor than rural ones

Figure 8

NEWS ITEM 6 (ECONOMIC SECTION)

Mathematical concept: Price increases, Percentages

Here is a piece about increase in air ticket prices during festive seasons. Which journey has risen the highest? By what percentage?

Why do the headlines say ‘demand surges but not supply’? Does this happen for other modes of travel?

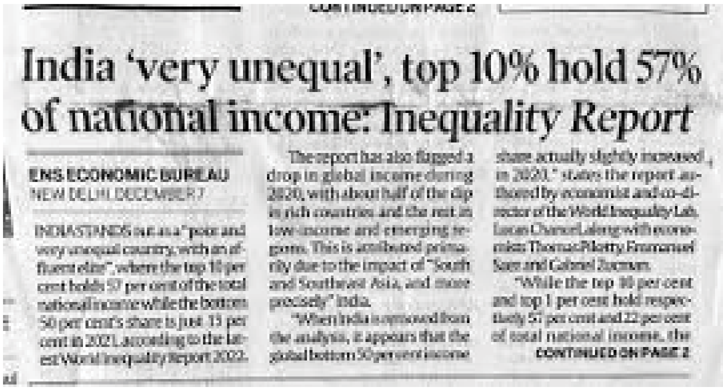
Do the prices of any goods rise during festive season? Which goods are these?

Airfares for Diwali skyrocket as demand surges but not supply

Figure 9

NEWS ITEM 7 (FINANCIAL SECTION)

Mathematical concept: Data comparison



India 'very unequal', top 10% hold 57% of national income: Inequality Report

ENS ECONOMIC BUREAU
NEW DELHI, DECEMBER 7

INDIA STANDS out as a "poor and very unequal country, with an affluent elite", where the top 10 per cent holds 57 per cent of the total national income while the bottom 50 per cent's share is just 13 per cent in 2021, according to the latest World Inequality Report 2022.

The report has also flagged a drop in global income during 2020, with about half of the dip in rich countries and the rest in low-income and emerging regions. This is attributed primarily due to the impact of "South and Southeast Asia, and more precisely" India.

"When India is removed from the analysis, it appears that the global bottom 50 per cent income share actually slightly increased in 2020," states the report authored by economist and co-director of the World Inequality Lab, Lucas Chancel, along with economists Thomas Piketty, Emmanuel Saez and Gabriel Zucman.

"While the top 10 per cent and top 1 per cent hold respectively 57 per cent and 22 per cent of total national income, the

CONTINUED ON PAGE 2

Figure 10

Here is a report from a newspaper in 2021.

What is the basis of the statement 'very unequal' ? How does the writer justify it? What does it imply?

Here is another chart depicting the share of wealth held by people.

Teachers can discuss the graph to see if the students are able to understand where the figure 77.4% has come from.

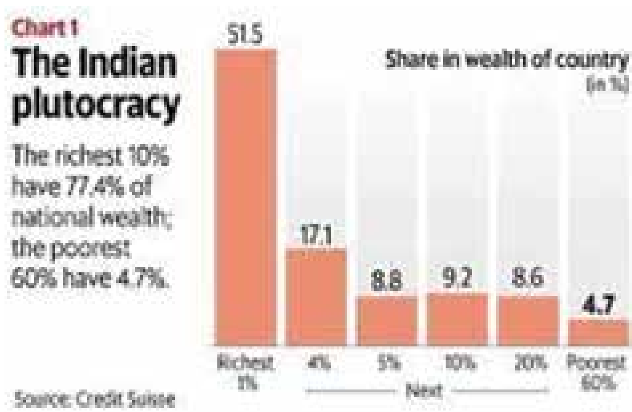


Figure 11

Here is a graph of rainfall data of Bangalore city.

For the given graph can the students build a table filling *x* and *y* values?

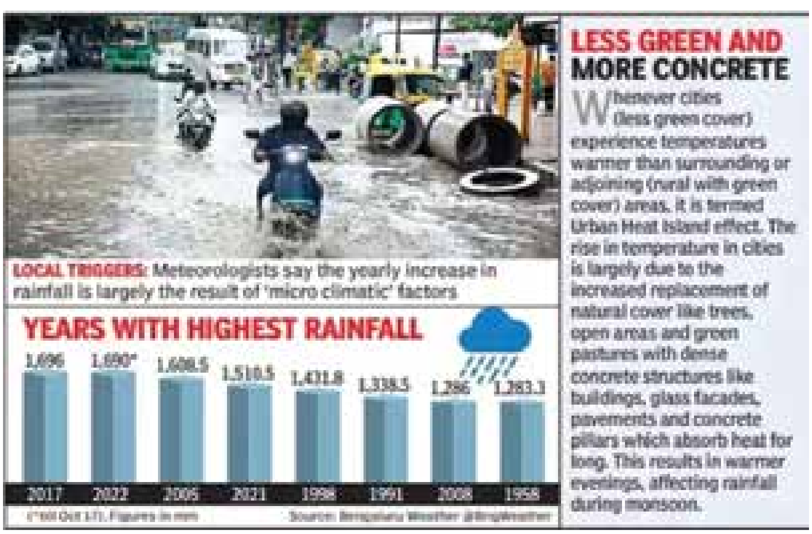


Figure 12

NEWS ITEM 8 (CURRENCY/ BULLION INFORMATION)

Math concept: Currency, currency conversions, financial literacy, direct proportion, inverse proportion

Item from the news 'Rupee falls 32 paise to close at 82.24 against US dollar'

Here is an opportunity for the teacher to explain the concept of different currencies and the conversion rates of one currency in terms of another. Students can find out currency used in different countries and the conversion rate of rupee to those currencies.

Students can watch for trends of price increase and decrease in prices of gold, silver etc. Teacher can explain about the economic importance of gold as it can be a means of exchange even if the currency collapses.

Figure 13

NEWS ITEM 9 (ADVERTISEMENTS FOR CONSUMER PRODUCTS)

Mathematics concepts: Prices, Taxes, Discounts

Full page advertisements offering goods at reduced prices appear in the newspapers.

Students can compare the original price and the reduced price to figure out the discount percentage.

On what kind of items is the discount percentage the highest?

Teachers can discuss GST and the students can compute the tax that is to be paid for the list of items.

Have a discussion around offers like 'buy two and get one' free. Do such promotions make us buy more things than we need?

Figure 14

NEWS ITEM 10 (FINANCIAL ADVERTISEMENT)

Mathematical concepts: Loans, Interest rates, EMI, down payment

Understanding loans for cars, trucks, homes, schooling, or other purposes

Figure 15

What does it mean to say starting @6.60%? What is an interest rate?

Figure 16

What does EMI stand for? What is a cashback? How does an EMI option benefit the consumer?

If the price of a Tv set is 55,000 how long will a consumer need to pay EMI if he choses to pay Rs. 2990?

For what kind of purchases do people take loans? What is down payment?

If the car cost 3.5 lakhs, how much interest would you need to pay in the first year, if your down payment is 50,000?

If your family plans to buy an apartment for 60,00,000 and pay 4,00,000 as down payment, how much interest would they need to pay in the first year?

NEWS ITEM 11 (ARTICLE)

Mathematical concepts: Interpretation of data


IMD: 2023 will not end as a drought year like '14 & '15

DRIEST AND HOTTEST AUGUST IN INDIA

Figure 17

On what basis is the statement 'driest and hottest august in India' being made? What data supports the statement? Has the situation between 2009 and 2023 deteriorated steadily or has it fluctuated?

By how much has the temperature increased from 2021?

Figure 18

In which part of India was the deficit in rainfall the worst? Which part of India did not experience a deficit?

Here is one more such data chart. How does this information compare with the regional data given in the previous data chart? If you consider the three months together, are there other parts of India that did not experience a deficit?

Environmental news is of great value in the classroom to discuss various environmental problems that are occurring in the world today. It can lead to discussions on carbon emissions, deforestation, glacial melting, global warming, erratic weather patterns. Etc.


metro hindustantimes

Planting 16,123 trees helped clear 3,420 tonnes of CO2

Figure 19

NEWS ITEM 12 (ANALYSIS OF A PAPER)

Mathematical concept: Areas/ Fractions/ Percentages/data handling

How much of the newspaper is really news?

Teachers can discuss with students what is news. Once the word news is clearly defined and commonly understood, students can identify all items which do not fall into the category of news. Advertisements, classifieds, etc.

Students can use various measures to measure the news component in terms of area or fraction or percentage to answer the question.

How can they estimate the space occupied by advertisements? Classifieds?

Is there more than one way of making these estimations?

Some students may approach it through fractions, some through estimation of area. Some may use transparent square grids to cover those sections and check.

They can finally express the answer as a percentage.

Different groups of students can study different


Chennai Times
FURNITURE FAIR
INTERNATIONAL DECOR EXHIBITION

Figure 20

papers for this purpose and come up with a comparative study.

It can lead to a discussion on the interdependence of newspapers and advertisers.

NEWS ITEM 13

Mathematical concept: Estimation, problem solving strategies, implicit idea of average.

Can the students estimate the number of words of a full page?

How would they go about it? Teachers can brainstorm various ideas before students begin to work on the problem.

Would it work if they figure out for a quarter or one eighth of the paper and multiply by the appropriate factor?

Would they use the columns of the paper as a guideline? Are all the columns of the same length?

Would they need to look at the headlines separately?

What would they do with the photographs and advertisements in the paper?

Figure 21

NEWSPAPER PROJECT

Math concept: Measurement, usage of model, Layouts, proportion

It is great fun to get students to design a single page newspaper.

The newspaper could incorporate numerical fun facts about themselves or interesting activities in the school.

They could do a statistical survey of students' favourite snack items.

Develop a timeline of their school history.

What kind of an advertisement would they design to encourage students to buy a healthy snack?

While designing and developing a newspaper students will need to make close observations of a model paper to make some choices.

One choice will be about the font size.

A closer look at any paper will make the students notice the varied sizes of letters.

Let students use the front page of one newspaper and make a study of the font (character/letter).

How many different sizes of fonts do they see? What are those sizes? In what range do they lie?

Figure 22

Typically, there will be three or four sizes.

Students can come up with their own names for these various sizes, say, Headlines, column heading and text size.

Question: If you had to design your own paper which has a headline, three column heads and text, how much space will you need to allocate for the headlines, the column headings and the text?

Indian Express

INDIA PROCLAIMED SOVEREIGN DEMOCRATIC REPUBLIC

Madras Turns Into 'City Of Lights'

FESTIVE ILLUMINATIONS ON REPUBLIC DAY

RAJENDRA PRASAD SWORN IN AS PRESIDENT

Proclamation Read Out By Retiring Governor-General

A Million People Watch Presidential Procession

Figure 23

Can the students come up with a reasonable answer? There is no single correct answer.

Students might carefully investigate the space occupied by the different categories in a paper and use that as a model.

They may use the font size of the headline, divide the width of the paper by the size to arrive at an answer. They may work out the number of columns and the space needed for the column heads.

The problem combines visual organisation along with numerical calculations.

What else do they notice about the fonts? Boldness, colour, etc.

Can they give some reasons for having different coloured fonts in a newspaper?

Can they think of reasons for having short headlines?

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PADMAPRIYA SHIRALI

PADMAPRIYA SHIRALI is part of the Community Math Centre based in Sahyadri School (Pune) and Rishi Valley (AP), where she has worked since 1983, teaching a variety of subjects – mathematics, computer applications, geography, economics, environmental studies and Telugu. In the 1990s, she worked closely with the late Shri P K Srinivasan. She was part of the team that created the multigrade elementary learning programme of the Rishi Valley Rural Centre, known as 'School in a Box.' Padmapriya may be contacted at padmapriya.shirali@gmail.com.

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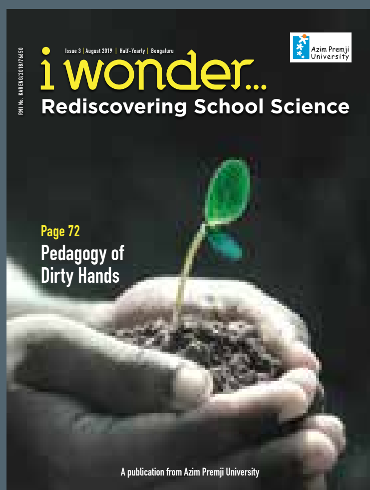

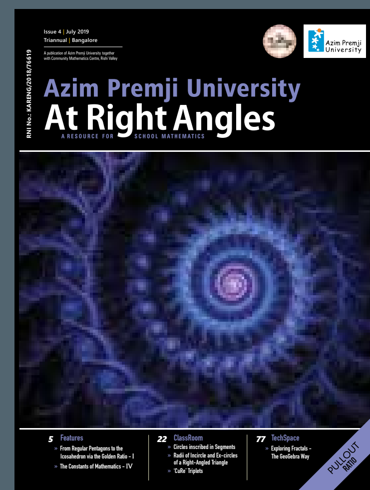

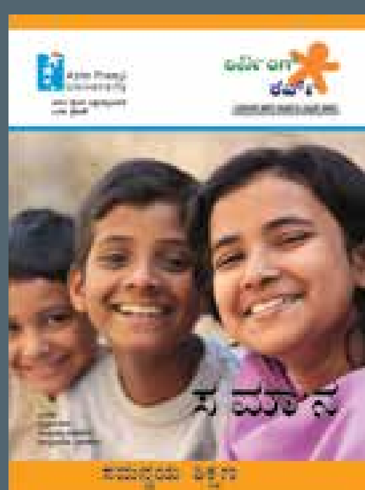

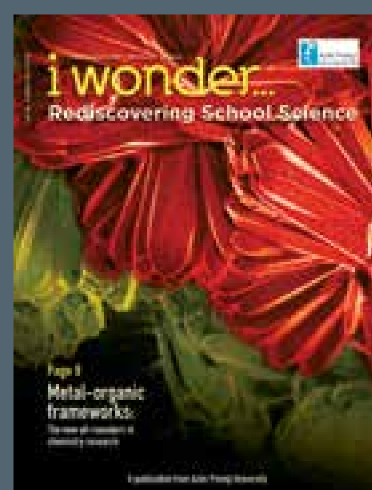

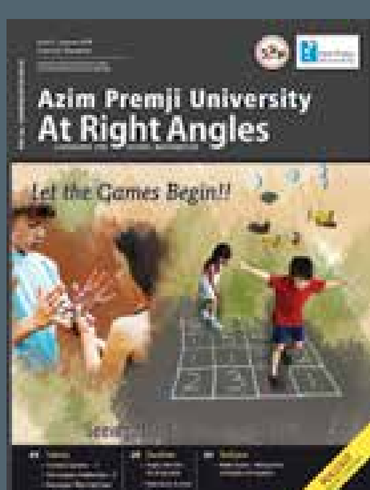

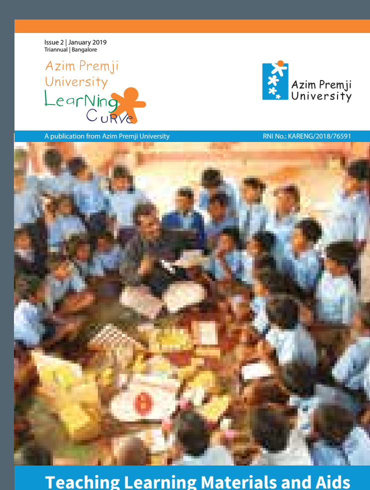

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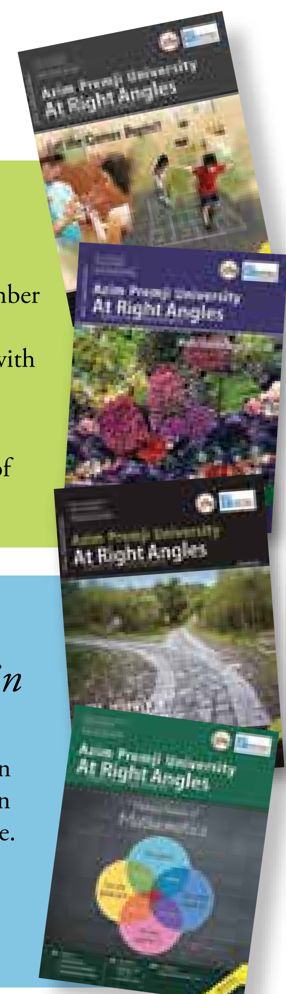

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Survey No. 66, Burugunte Village, Bikkanahalli Main Road, Sarjapura, Bengaluru – 562 125

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