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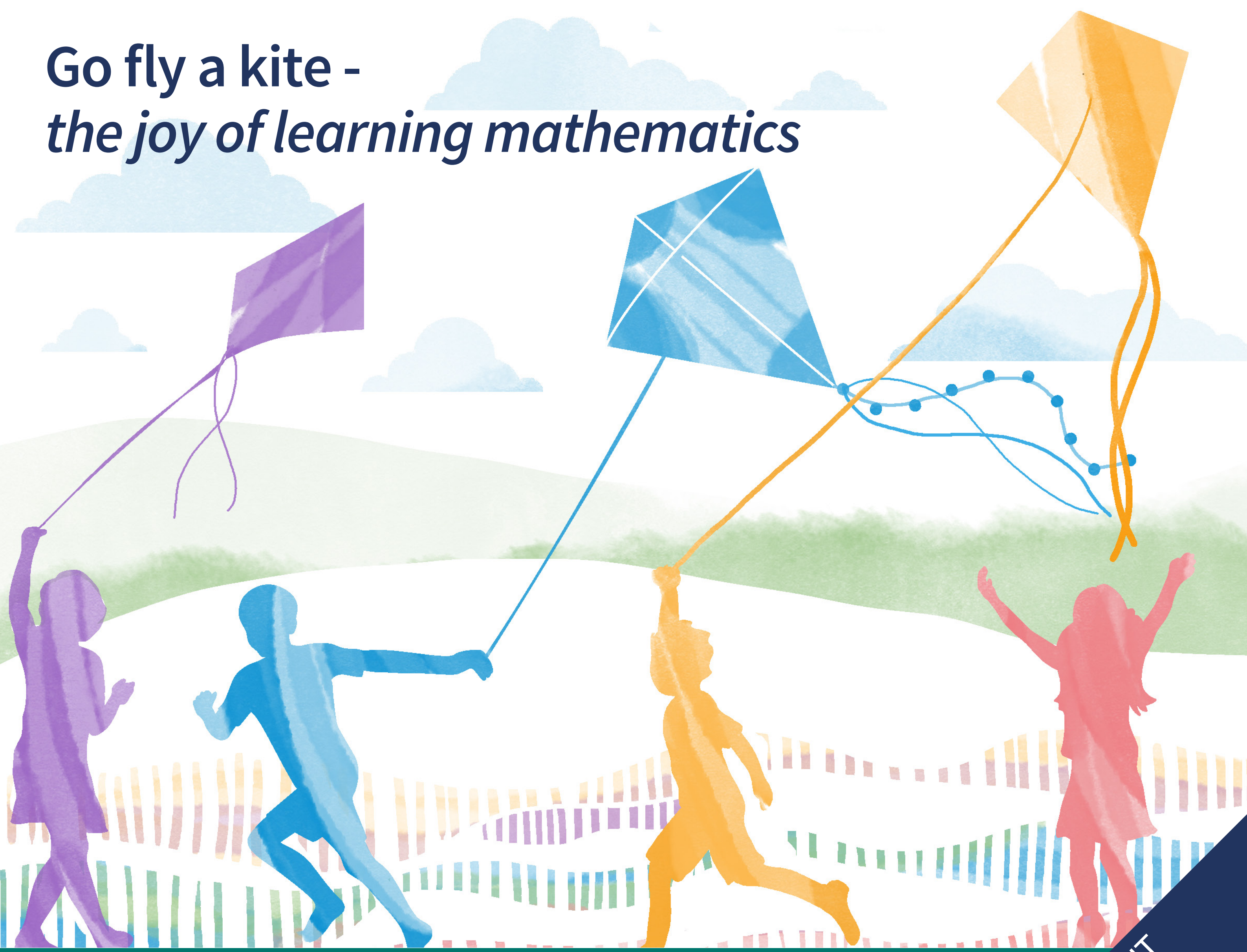
Azim Premji University At Right Angles

A RESOURCE FOR SCHOOL MATHEMATICS

ISSN 2582-1873

Go fly a kite - *the joy of learning mathematics*



PULLOUT
AS HOT AS MATH
AS COOL AS MATH

The joy of mathematics – A phrase that has been used in the National Curriculum Framework – has to be handled mindfully and explored in depth. What is the joy of mathematics? Is it just about fun classrooms and frivolous mathematics? Can we find joy in the rigour of mathematics? Isn't that an oxymoron?

While At Right Angles cannot prescribe a dose of joy, we certainly can showcase ways in which to infuse students with the sheer exhilaration of seeing mathematics everywhere, of seeing patterns in shapes and numbers, of making connections and testing generalisations, of constructing sensible arguments, of validating their stance. Let's release our inner child with confident and fearless minds.

From the Editor's Desk . . .

Here we are at the second issue of the new At Right Angles, where we consciously attempt to connect with pedagogical issues associated with teaching mathematics at the primary school level. A big thank you to all our authors who have shared their experiences and learning and to our readers who have reached out in appreciation and provided valuable critique.

In the **Features** section of the July 2024 issue, we continue to explore the multi-disciplinary approach with Rima Kaur and Manisha Goyal. Shared reading experiences have been found to be a powerful tool in the classroom. Here, they explore creating a Big Book to explore the idea of handling data from the child's context. And what could be better than talking about such explorations? Prachi M makes a powerful argument for *Mathematical Discourse* in the same section.

Azim Premji University alumnus Aruni Joseph sets the ball rolling in the **ClassRoom** section with an account of her experiences in teaching the concept of angles during her field internship. James Metz and Brad Uy explore interconnections between algebra and geometry in *Why 6*? – a question that uses hands on activities with interlocking cubes to explore an algebraic identity. Visualisation is also invoked in the article by Math Space on a better algorithm for Multiplication and in the review of Algebra Tiles.

The **Joy of Mathematics** section continues the *Guesstimation* article with classroom activities to practise this skill – I am quite sure that these will infect students' conversations at home and during play! And find your *Special Year*- when will the last two digits of your birth year be your age? I have a Special Year coming up soon- but why can't it be in 2025? Read on to find out.

Our cover this time is inspired by the **TearOut** on *Kite Families*. Teachers are encouraged to put up this beautiful poster in their classrooms and to use the questions provided - and others which their students may come up with- to take their students on a magical quadrilateral journey. Math Space has provided some guidelines to keep the discussions grounded in mathematical rigour – so your seat belt has been fastened!

Padmapriya Shirali's **PullOuts** have always been in demand, in this one she turns up the heat with explorations on *Temperature related activities* which can spark discussions on climate change, sustainability and conservation of resources. We end with a craft activity – making a Temperature Quilt. What more could a teacher need for the multi-disciplinary approach!

We are delighted to announce that articles at the higher level are now back in the magazine, in the **online version** of At Right Angles. Scan the QR codes given on the Contents Page to read student Aneesh Kumar's account of the *First Sine Table*, written in collaboration with Dr. Vijay Singh. And top this up with a tech investigation of *Poisoned Samples* – Kumar Gandharv Mishra teaches you how to simulate a lab investigation using Excel. Best of all, the planning of the investigation develops your computational thinking skills.

In order to stay connected to those who will be using At Right Angles in their work, we have designed a questionnaire and request our readers to mail in at AtRightAngles.editor@apu.edu.in in order to get the survey link. Happy reading! And do keep in touch.

Sneha Titus
Editor, At Right Angles.

At **Right Angles** is a publication of Azim Premji University. It aims to reach out to teachers, teacher educators, students and those who are passionate about mathematics. It provides a platform for the expression of varied opinions and perspectives and encourages new and informed positions, thought-provoking points of view and stories of innovation. The approach is a balance between being an 'academic' and 'practitioner' oriented magazine.

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Contents

Features

Our premiere section presents a diverse range of articles centred around mathematics education. The range varies from discussions on various aspects of mathematics classroom practice to critical analysis of textbooks. Written by practising mathematics educators, the common thread is sharing critical perspectives.

ClassRoom

Articles in this section are helpful in designing and implementing effective instruction. They draw directly on personal experiences of teaching with reflections on both successful plans as well as opportunities and ideas for improvement. They strengthen and support the teachers' own understanding of these topics and strengthen their pedagogical content knowledge.

The Joy of Mathematics

This is a section that simply celebrates the joy and beauty of mathematics. You will find light anecdotes, comic strips, cartoons, essays and behind all of these the beautiful reasoning that amplifies the nature of mathematics.

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Continue . . .

Review

Our Review section offers a diverse range of insights on educational resources in mathematics. We examine a variety of resources: textbooks, books on mathematics and mathematics education, teaching-learning materials, interesting websites, and educational games and software. We feature evaluations not only from experienced educators and subject experts but also welcome perspectives from practitioners and enthusiasts in the field. This inclusive approach ensures a rich variety of viewpoints, providing our readers with comprehensive and accessible reviews that can inform their choices in educational materials.

TearOut

The TearOut is designed to be of immediate use to the teacher. It could be a poster to be put up in the classroom to motivate observation and discussion, or it could be a worksheet to be photocopied and shared. It is always accompanied by questions and guidance for the teacher.

PullOut

The PullOut takes a hands-on, activity-based approach to the teaching of the basic concepts in mathematics. This section deals with common misconceptions and how to address them, manipulatives and how to use them to maximize student understanding and mathematical skill development; and, best of all, how to incorporate writing and documentation skills into activity-based learning. The PullOut is theme-based and, as its name suggests, can be used separately from the main magazine.

Online Articles

Using Big Books To Teach Data Handling in the Foundational and Preparatory Stages

MANISHA GOYAL &
RIMA KAUR

‘Big Books’ and shared reading experiences

‘Big Books’ are oversized story books, often containing large pictures and fewer words in large print. This allows for multiple readers to access it and read together comfortably. In the past few decades, the use of big books has gained prominence for ‘shared reading’ experiences with young children in the classroom (Karges-Bone, 1992) (3).

Figure 1: A shared reading experience.

Keywords: *Foundational Literacy and Numeracy, Foundational Stage, Preparatory Stage, early childhood education, data handling, big books, shared reading, integration of disciplines*

Classroom Vignette 1

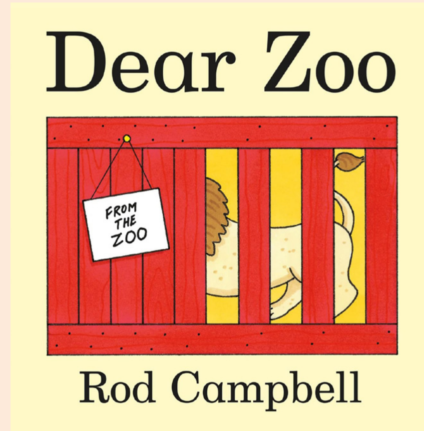


Figure 2: 'Dear Zoo' by Rod Campbell, a 'lift-the-flap' interactive story book, which is also available in the form of a big book.

Picture a group of class 2 children huddled around their teacher, looking over at a large story book gently resting on the teacher's knee. The cover page says '*Dear Zoo*'. The teacher, Rini, draws children's attention to the cover page, where you see a partially hidden lion in a crate. After talking about the title and picture for some time, Rini flips to the first page and reads, "*I wrote to the zoo to send me a pet.*" Most of the page is occupied by a large crate that says 'VERY HEAVY!'. Children try to guess which heavy animal can be kept as a pet. A few say elephant, while others say hippo! Rini lifts the flap - it is an elephant! She reads further, "*They sent me an… elephant! He was too big! I sent him back.*" Rini asks questions that make children wonder - *Where would we keep an elephant? What would it eat?* And the story continues… The subsequent pictures partially reveal more animals - a giraffe, a lion, a camel, a snake, a monkey, a frog, and finally a puppy! Each time, Rini lifts the flap by herself or asks someone to lift it for her. A few children disagree about having a puppy as a pet - *A monkey is much more FUN!*

The big book is then used by Rini for a number of post-reading activities such as - talking about which pet children would like to keep for themselves, telling them more animal stories, making masks and role-playing different animals, and writing a letter to the zoo and requesting for a pet of their own.

Rini frequently uses big books for creating shared reading experiences in her classroom to develop children's literacy skills as illustrated above. But these days, she is also on the lookout for story resources that can enrich her teaching in other subjects. One day, she comes across another storybook - *What Would You Like To Drink?* by Anand Janardhan (illustrated by Ayesha Punjabi) (Janardhan, 2021) (2). The story is about a young girl, Tara, who helps her Dada (grandfather) remember the beverage requests of 15+ people who visit their home for a birthday party. The story embeds the concept of collection and organization of data in an everyday problem context. Rini reads the story and gets a brilliant idea for her mathematics classroom!

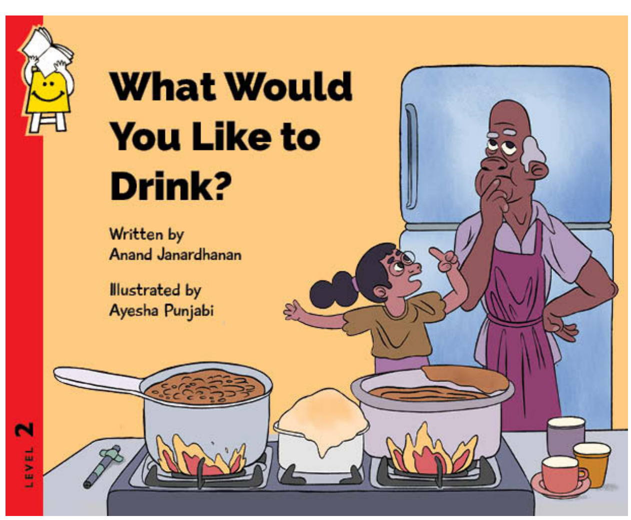


Figure 3: 'What Would You Like To Drink?' by Anand Janardhan (illustrated by Ayesha Punjabi) (Janardhan, 2021) (2), a story embedding the grouping and counting of data in an everyday context.

Classroom Vignette 2

Rini prints this story on A3 paper with a large font and decides to use it as a big book. She uses the whiteboard stand to prop the big book and uses a wooden ruler for pointing. She conducts a shared reading session using the big book and asks questions to spark children's mathematical thinking. When Dada is puzzled about the quantity (number of glasses) of each beverage he needs to prepare, Tara offers to help. Rini asks, "*How do you think Tara will help Dada*?" A few children respond, "*Tara will call out the name of each beverage one by one and ask the guests to raise their hands. She will count the number of raised hands and tell Dada about it.*" One child says, "*Tara will ask the guests to stand in different corners based on their beverage choice.*" A rather unique response from another child, "*Tara can take down beverage orders from each guest, just like a server does in a restaurant.*" Rini is thrilled to hear such a variety of responses! This shows her that many children in her class have a readiness to formal introduction to data handling techniques. Rini goes ahead and flips to the next two pages. Before reading the story, she asks children to look at the pictures and describe Tara's solution. With Rini's help, the children infer that Tara first lists the different drinks - tea, coffee, juice, and milk in one column of the table. She then asks each guest what they want and marks their choice one by one in the next column of the table. Rini then completes the story and asks, "*What do you think of Tara's solution? How is it different from what you were suggesting? What are the similarities? Do you think this is a useful idea for other problems we face in our lives?*" Afterwards, Rini also encourages children to look at the table and share their observations about the guests' beverage preferences. For example, which was the most liked beverage, which one was ordered the least, predicting/estimating which ingredient Dada would need the most, etc.

Here are the follow-up activities that Rini does in her classroom:

- Rini asks, "*What kind of beverages would you like to have if we have a get together in our classroom?*". From children's responses, preferences such as *chaas*, *bel* sherbet, tea, and *nimbu pani* emerge, which are also popularly consumed locally.
- To decide the quantity (number of glasses) of each beverage, Rini asks if they would like to create a table like Tara did to help Dada. Rini first lists the four beverages. She invites one child, Kaushal, to come forward and populate the rest of the table. Just like Tara did in the storybook, Kaushal asks each child what they want and marks their choice one by one in the table. This leads to the creation of Chart 1:

Chaas	***Bel* sherbet**	**Tea**	**Nimbu pani**
✓✓	✓✓	✓✓	✓✓
✓✓	✓✓	✓	✓✓
✓✓	✓✓		✓✓
✓✓			✓✓
			✓✓
			✓
8	6	3	11

Chart 1: Table recording the beverage preferences of children in Rini's class

- Rini and the children then decide to make a big book of their own. They substitute the birthday party with a classroom party and add their preferred beverages. The story begins, '*Once upon a time in Posaliya town of Sirohi district in Rajasthan, 27 children of class 4 decided to have a party…*' Rini keeps the big book in the reading corner and notices children reaching out to it and reading it in small groups.
- After this experience, Rini promises that they will have a classroom party coming Saturday where everyone's preferred beverages will be prepared together and served!

In the days that follow, Rini uses contexts given in the textbook (NCERT Mathematics textbook, Class 2, Chapter 11: Data Handling) (NCERT, 2023) (9) to further develop their data handling skills as per the curricular expectations for this grade level. She first guides the children to understand the contexts by observing and describing the given pictures. This is followed by children counting and collecting data for given categories e.g., favourite colour, type of fruit/vehicle, etc., recording their findings in the form of tables, representing their data in the form of simple pictographs, and talking about the findings by looking at the pictographs.

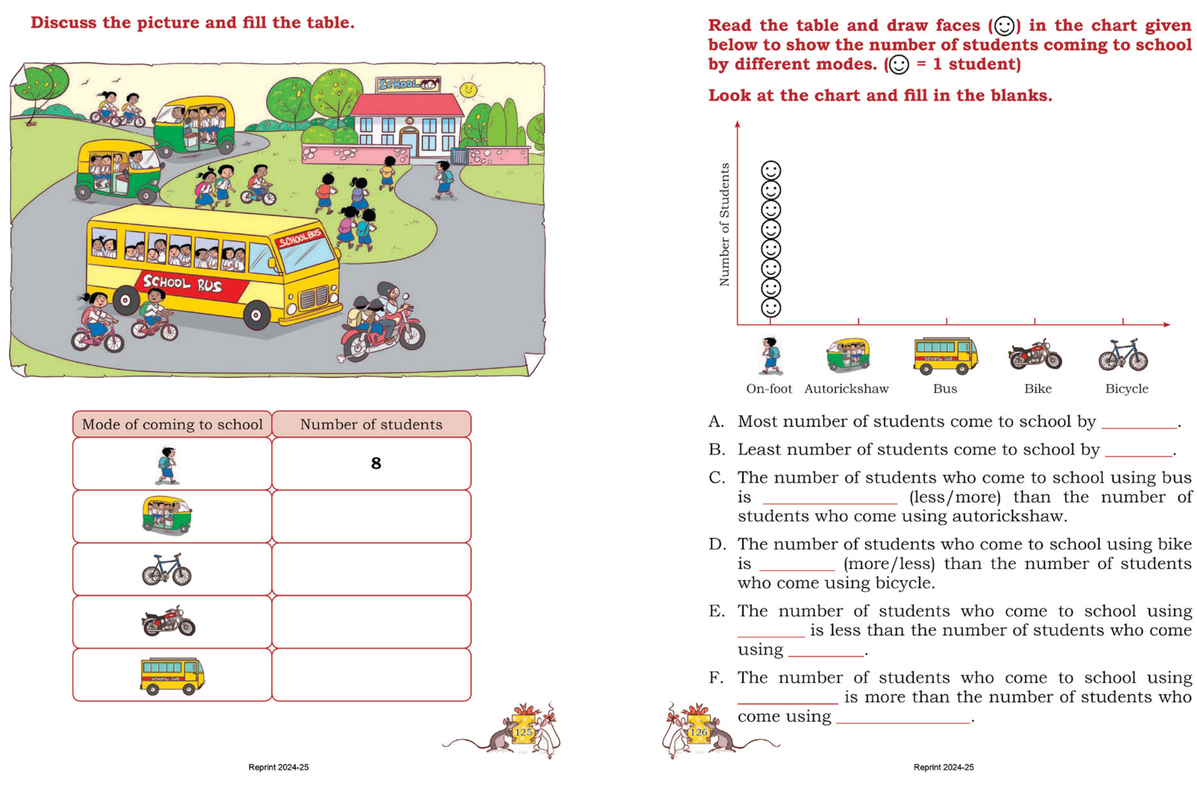


Figure 4: Source: NCERT Mathematics textbook, Class 2, Chapter 11: Data Handling, pp. 125-126 (NCERT, 2023) (9)

Rini then culminates the topic of data handling in her class by setting a task where children have to create a pictograph for the beverage table they had created earlier. Children create the following pictograph and attach it to their big book:

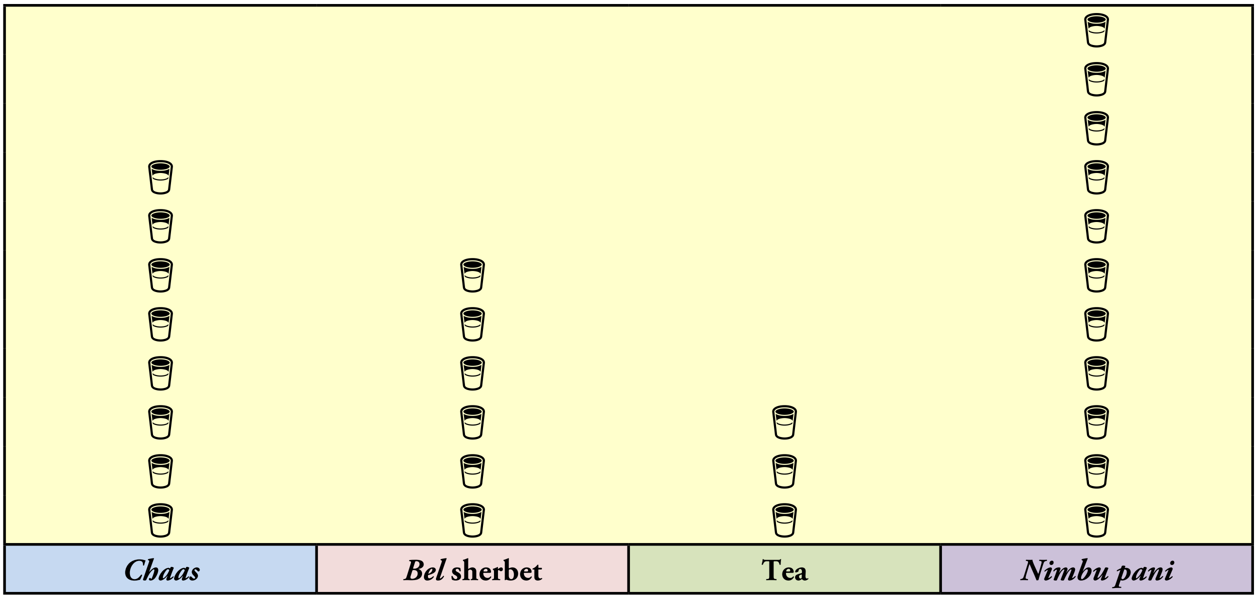


Chart 2: Pictograph representing the beverage preferences of children in Rini's class, based on Chart 1

Using big books in teaching mathematics

Rini's experience is an example of an intimate and participatory reading experience for introducing a mathematical concept. Though such experiences are generally recommended for language and literacy development[1], there is a growing acceptance and practice of embedding mathematical concepts within narratives such as stories and rhymes for young children (NCF-FS 2022, p. 141) (7). Such integrative approaches support and extend the conceptual understanding of complex mathematical development. This is because familiar contexts allow children to draw on their prior knowledge to bring meaning to mathematics. Integrating mathematics and literature creates opportunities to introduce new vocabulary, to make connections among abstract concepts, and showcase ways that mathematics applies across the curriculum. (Koellner et al, 2009) (4). We saw that Rini's children offered a variety of solutions for Tara's Dada. This was because they had become immersed in solving a problem in a familiar context. Young children are known to choose a variety of ways for representing their data when given the opportunity, such as by making templates/drawings, simple pictographs and bar graphs, making block charts using sticky notes, etc. (English, 2013) (5). Here the use of a simple story provided such an opportunity to children.

A cursory survey of publishers of children's literature in the market reveals a surge of stories where mathematical concepts are clearly embedded. These are usually referred to as 'STEM' (Science-Technology-Engineering-Mathematics) books. In the early years, they usually embed concepts related to EVS and mathematics such as plants, health and hygiene, birds and animals, mathematical operations, shapes and objects, measurement, time, etc., in their plotlines. Here are a few examples from popular publishers of children's literature in India:

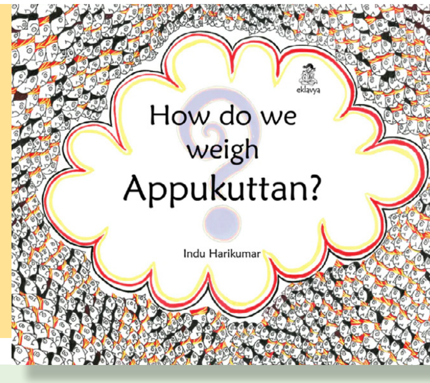
[1] For example, to develop interest in stories, build familiarity with print, demonstrate a variety of reading strategies, expand vocabulary, and develop early reading skills such as recognizing words and developing connections between sounds and letters.

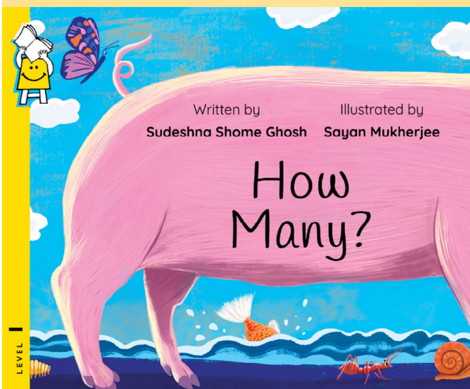
How Do We Weigh Appukuttan?

(Eklavya)

Written by Anjali Alappat and illustrated by Yogee Chandrasekaran

Concept covered: Weight (standard and non-standard units)





How Many?

(Pratham Books)

Written by Sudeshna Shome Ghosh and illustrated by Sayan Mukherjee

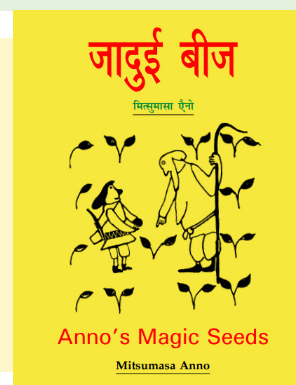
Concept covered: Counting

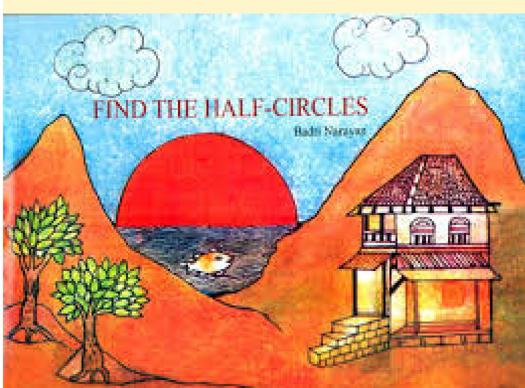
Anno's Magic Seeds

(Bharat Gyan Vigyan Samiti)

Written by Mitsumasa Anno and Hindi translation by Arvind Gupta

Concepts covered: Addition and multiplication, can be extended to arithmetic and geometric progression too (for higher grades)





Find The Half-Circles

(National Book Trust)

Written by Badri Narayan

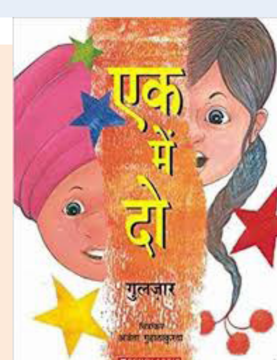
Concept covered: Shapes (semi-circles)

Ek Mein Do

(Translation: *Two In One*) (Scholastic)

Written by Gulzar and illustrated by Anjana Guha Thakurta

Concept covered: Counting

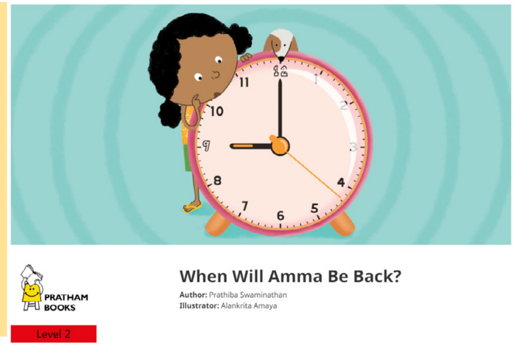


When Will Amma Be Back?

(Pratham Books)

Written by Prathiba Swaminathan and illustrated by Alankrita Amaya

Concept covered: Measurement (time)



Keshav's Kolam

(Karadi Tales)

Writte by Shobha Viswanath and illustrated by Leeza John

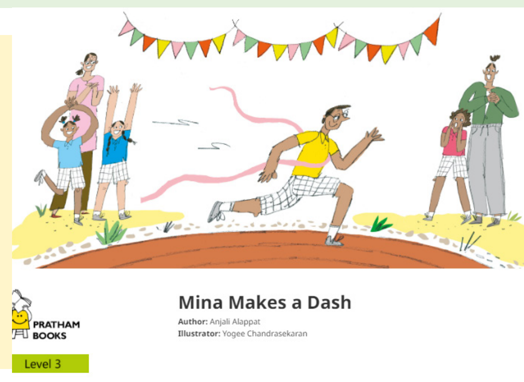
Concept covered: Patterns, symmetry

Mina Makes A Dash

(Pratham Books)

Written by Anjali Alappat and illustrated by Yogee Chandrasekaran

Concept covered: Measurement (length)



Gola Gola

(Pratham Books)

Written and illustrated by Aithihya Ashok Kumar

Concept covered: Venn diagrams

The Animal Plot

(Pratham Books)

Written by Lokesh Khodke and illustrated by Lokesh Khodke

Concept covered: Bar graphs

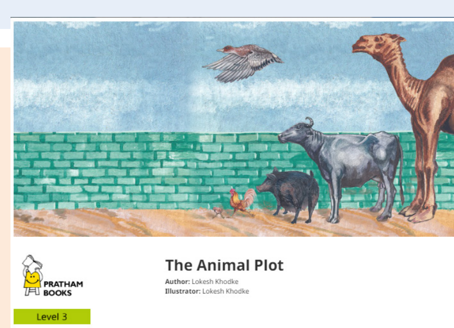


Figure 5: Samples of children's literature in India.

Such children's literature can prove to be an excellent resource for teachers to include in their mathematics classroom in a variety of ways. Teachers can also look for the potential to engage with mathematical ideas in non-STEM children's literature. For example, even classic tales like *Jack and the Beanstalk* or *The Monkey and the Cap Seller* can be adapted and used for teaching various concepts such as counting, addition, subtraction, measurement, comparisons, etc., as per the teacher's imagination. The new mathematics textbooks by NCERT also have ample contexts for building mathematical stories. The use of stories also provides a seamless opportunity to engage children in developing problem solving skills, which is also an important goal of teaching in a mathematics classroom (NCF-SE 2023, p. 177) (8).

Data handling

Data handling is a major component or area of mathematics learning in the Foundational Stage along with number and its relations, basic mathematical operations, shapes and spatial understanding, patterns, and measurement. In today's world, with the constant bombardment of huge amounts of data, bar graphs, pie charts, etc., have become routine forms of communication in almost all walks of life. Therefore, a sound understanding of data handling is an important component of children's mathematics education (Shirali, 2016) (10). The National Curriculum Framework for Foundational Stage (NCF-FS 2022) defines data handling as '*understanding the collection of data, collecting and analyzing it*' (NCF-FS 2022, p. 121) (7). It further states that data handling in the Foundational Stage involves sorting, classifying, and counting objects in groups (NCF-FS 2022, p. 333) (7).

Children naturally group and count items outside school in a variety of real-life contexts such as while playing with their toys, laying the table, and cleaning up after themselves. When children first enter the formal school space, a few ways in which these experiences get extended is during attendance time, during a contest, and while choosing a game/sport to play.

When children develop sorting and classifying skills in early childhood through concrete experiences, they can be supported to represent the data using pictographs and block charts. Such experiences help children notice patterns in their surroundings and develop skills that can be applied for handling data in later stages of schooling. For example, in the Preparatory Stage, collecting, organizing, and representing data takes relatively more abstract forms such as using tally marks, bar graphs, and pie charts along with development of the ability to read and interpret/infer meaning out of these representations (Shirali, 2016) (10).

Gaps in children's understanding of data handling

As children become older and move from the Foundational Stage to the Preparatory Stage and later, they deal with mathematics in more abstract ways. Unfortunately, data handling is a neglected area in early mathematics. If children have not engaged with sorting and classifying information in concrete ways in the early years, dealing with abstract data in the forms of tables, charts, and graphs becomes even more challenging. For example, a child despite being in class 4 may look at Chart 3 which uses 'pictographs', and find it difficult to grasp the following:

- What each part of the chart represents e.g., that each green triangle represents 3 children, the title of the chart, and other data labels.
- That the number of children in each task can be calculated from the chart, along with the total number of children.
- That the ideas of 'more', 'less', and 'equal to' can be explored through the chart e.g., which task is being done by most children.
- That all this information can be represented in the same chart.

• That all this information is related to their daily lives, and that they too can collect and represent similar data from their surroundings.

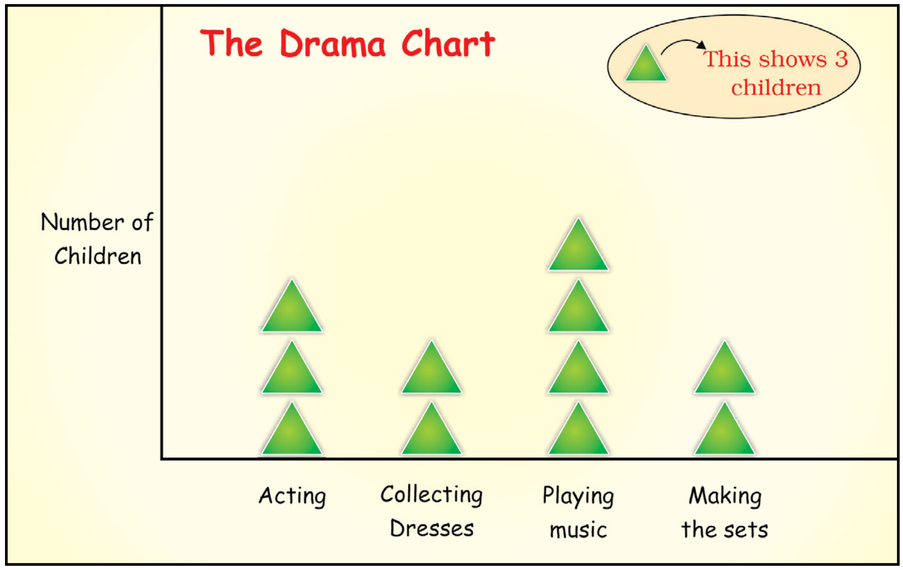


Chart 3: *Pictograph representing the different tasks being done by children of a class for a drama*

Source: *NCERT Mathematics textbook, Class 4, Chapter 14: Smart Charts, p. 163 (NCERT, 2007/2024) (6)*

As children move to higher grades, data is represented in more complex ways such as by using tally marks, bar graphs, and pie charts. Children may incorrectly group tally marks in sets of five, misinterpret graphs due to confusion regarding the use of scale, or misinterpret pie charts due to difficulty in understanding fractions.

Summary

As in the case of other mathematical concepts, misconceptions regarding data handling are typically due to how the content is presented and sequenced in the textbooks, along with gaps in the teacher's pedagogy, where children typically do not get a lot of hands-on experience of collecting, sorting, and representing data in authentic, meaningful situations. Use of children's literature can help in providing meaningful contexts for developing and practicing mathematical concepts in the classroom. More specifically, big books emerge as a developmentally appropriate tool for working with young learners as their large, visually engaging pages effectively illustrate data handling concepts and make the learning process interactive and accessible. When teachers adapt existing storybooks to make big books or co-create big books with children based on stories and conversations with them, rich and relevant contexts emerge for teaching not only mathematics but language and other subjects like The World Around Us (EVS) too. Children also love to repeatedly read big books in their own time, further serving the purpose of developing their interest in reading and making them keen readers. (Karges-Bone, 1992) (3).

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Manisha Goyal has worked as a Primary School teacher in both private and government schools in Delhi for 12 years. Alongside teaching, she also managed the library for her Primary School for about 5 years. She has worked towards developing her students' interest in literature as well as in mathematics. She has a Bachelor's degree in Elementary Education from Delhi University and a Master's degree in Education from Bharat Ratna Dr B. R. Ambedkar University, Delhi.
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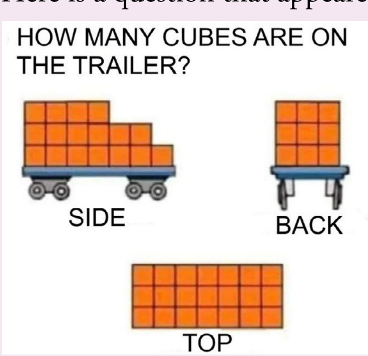
Rima Kaur is an Assistant Professor at the School of Continuing Education and University Resource Centre (SCE-URC), Azim Premji University, Bengaluru. She has a B Ed from Guru Gobind Singh Indraprastha University, Delhi, and an M A in Education from Bharat Ratna Dr B R Ambedkar University, Delhi. Her areas of interest are Early Language and Literacy and Early Childhood Education. Rima says that she was terrified of math in school and cannot believe that she has now written an article in a mathematics magazine.
She may be reached at rima.kaur@azimpremjifoundation.org

Opening the Question Up!

Here is a question that appeared on our WhatsApp group.



Is there only one correct answer to this question?

This is a great hands-on exploration even for students who haven't studied the concept of volume. In fact, better answers may come from them. Have fun experimenting with interlocking cubes, model the given information and unleash your creativity!

Send in your solutions to AtRightAngles.editor@apu.edu.in

Mathematical Discourse: Going Beyond Right Answers

PRACHI M

It was a winter day, and all the students gathered on the ground for assembly. The air was filled with the melodious notes of the national anthem and other uplifting tunes, setting the stage for a day of learning. As the last strains of music faded away, the teacher asked, "Who will tell me the table of 14?" Some students raised their hands while some started hiding behind others. "Let's start from Class 3, Aryan you tell me, what is 14 times 7?" Aryan seemed nervous and though he tried, could not recall what 14 × 7 was. After the assembly, the teacher proudly told me how some of their students could recite the tables till 30. Later, in a fourth-grade class, a student flawlessly solved double digit multiplication but confessed to simply following instructions when asked why he wrote a '0' while multiplying by the second digit. These two anecdotes highlighted for me, how school mathematics often focuses on memorization and question-answer conversations, stifling the joy and creativity inherent in exploring numbers.

What is mathematical discourse?

The communication that occurs in a mathematical classroom is known as mathematical discourse. In most classrooms 'correcting discourse' is mostly used, in which the communication stays limited to students answering the teacher's questions and the teacher saying whether the answer is correct or wrong. Only the prescribed method is followed by the students in order to arrive at the 'correct answer', leaving no scope to explore the process. However, the goals of mathematics education aim to

Keywords: *Communication, exploration, reasoning, understanding, courtesy.*

develop logical thinking, explaining patterns, making, refuting, and proving conjectures, problem-solving, computing fluently, and communicating clearly and precisely.

Good mathematical discourse happens when students share their ideas and listen to each other's mathematical thoughts instead of the teacher dominating the communication. This helps them understand mathematics better because they see different ways of thinking about problems. It also helps them feel more confident in their mathematics skills as they become better at understanding and explaining concepts. This helps teachers too as they can see what students understand and what they don't, so they know how to help them learn better.

My experience of setting mathematical discourse in a classroom

An attempt to promote mathematical discourse was made while working on number sense with elementary grades. Certain rules and norms such as listening to others, respecting others' opinions, justifying one's own opinion, etc. were set for our mathematics classroom. The key rule that was set while establishing the discourse was: **don't just say the right answer but tell us how you approached the problem.** Two strategies which were questioning and explaining mathematical thinking were widely used in this process. Different structures such as one-to-one conversations, working in pairs, working in groups, and whole-class discussions were used at different times. Students also used different representations and models while explaining their mathematical thinking.

While working with students of Class 1, it was initially difficult to get them to talk about what they were thinking. In the initial stage, we had good one-to-one communication. Later, they were also able to engage in communication during group work and whole class discussions. I noticed that the Class 3 students had the urge to either give the right answer or listen passively. So, I made it a practice to follow every answer with the question 'How did you get it?' Initially, I saw students struggling to express what they were thinking and end up giving one-word answers. At such times, asking more follow-up questions such as 'Can you explain this step?', 'What do you mean by this?' etc. really helped. At times reframing what they wanted to say using complete sentences and asking them if that was what they meant, also helped as students were able to edit their responses and clarify their thinking. After a few days of practice, I was delighted when the students started asking their classmates 'How?' they got their answers.

In the following examples, probing discourse in which the teacher asks different probing questions as mentioned above, is used to understand student thinking and approach. This type of discourse helps support procedural fluency and conceptual understanding.

One-to-one communication:

While working on addition with Class 1, students were asked to draw different representations to explain their answers to the question:

'Ramesh has 5 jamun trees, 3 mango trees, and 7 guava trees in his orchard. How many trees are there in the orchard?'

How could you write/draw what you are doing?

Ananya's representation of the problem is shown in Figure 1.

Teacher: How did you count?

Ananya: I counted one, two, three……fifteen

Figure 1

Immediately the teacher understood that she used the 'count-all' strategy.

It helps to understand exactly at what level the student is on the learning trajectory and assist them to move upward on it. Following this, we worked on different techniques to help students use more efficient ways of approaching such problems by using a 'count-on' strategy.

Communication during pair work:

Students asked their partners different word problems which they had framed themselves.

Aryan: If I have 30 chocolates and I gave 6 to Virat, how many chocolates will I have?

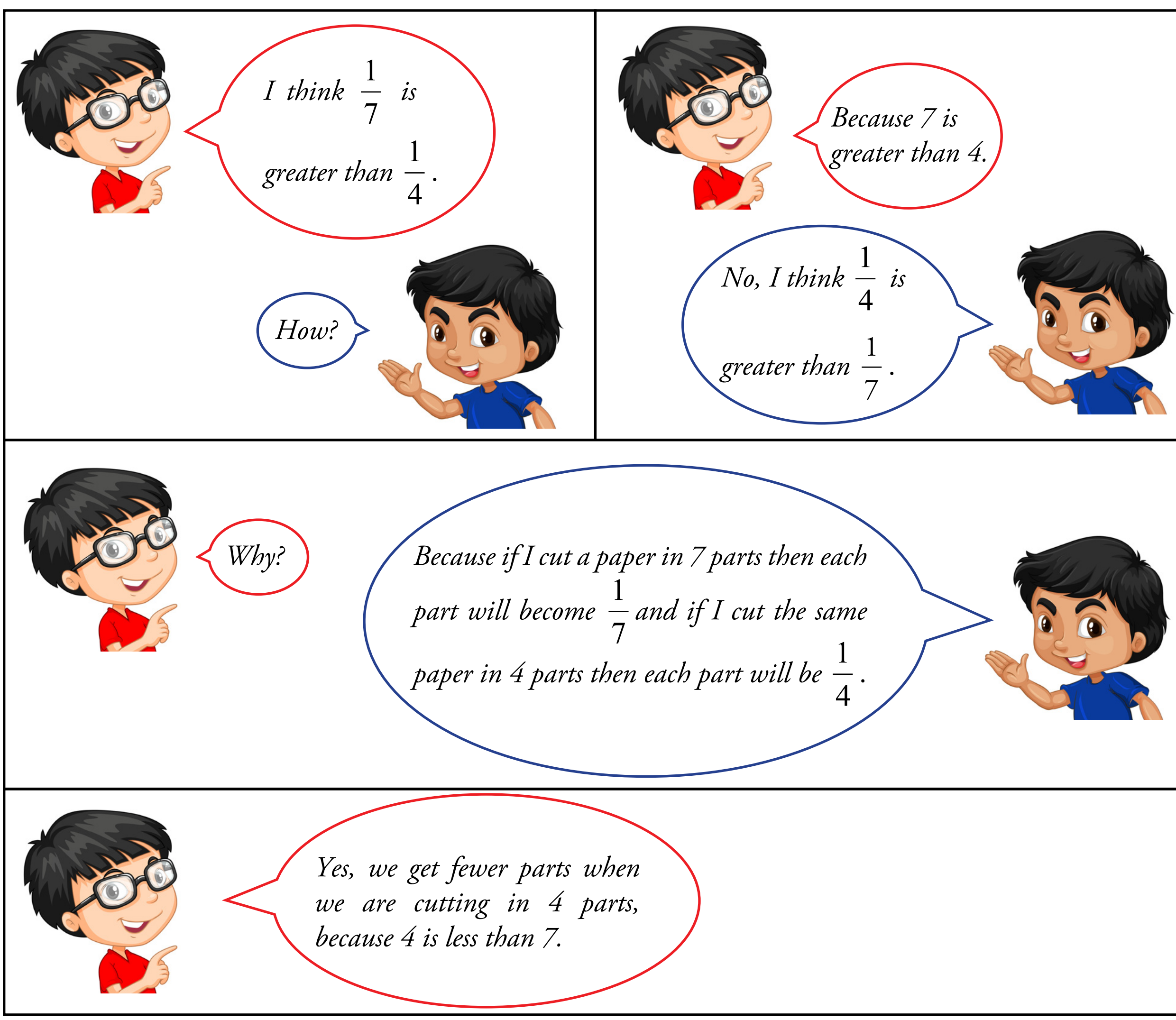
Harish: 22... (after some thinking) No no it will be 24.

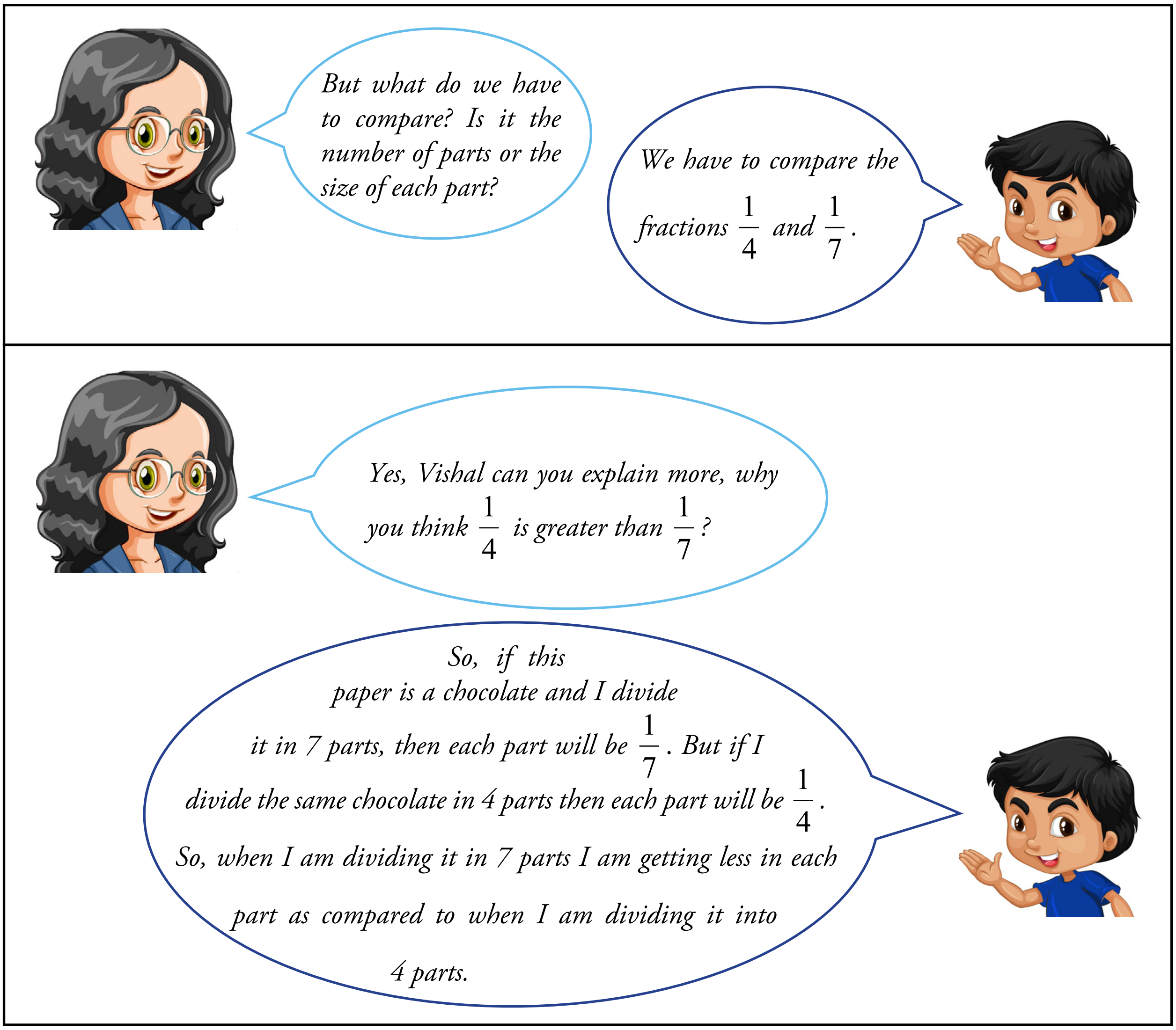
Aryan: How?

Harish: 6 and 4 is 10. So, if I gave 6 to Virat, I would have 4. And, I have 20 of 30 chocolates, so I will have 20 and 4...24 chocolates.

The above dialogue helped the student to refine his thinking and confidently express his mathematical strategy.

Question: Which fraction is greater: $\frac{1}{7}$ or $\frac{1}{4}$?





Such discourse helps to identify students' misconceptions which otherwise go unaddressed. It also helps in developing reasoning, better understanding and confidence in communication.

Communication during whole class discussion:

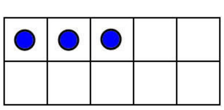
1) 

Figure 2

Teacher: What can you see? (Before this task we worked on ten bonds)

Ashu: I can see 3.

Teacher: Yes, there are 3 blue dots. What else can you see?

Soumya: I can see 3 + 2 + 5 = 10

Teacher: How Soumya? Can you explain more?

Soumya: Because there are 3 dots and if we add 2 more, it will be 5 and the remaining 5 would make 10.

Teacher: Yes, so Soumya is saying that if she adds 3 and 2 it will be 5 and adding again 5 will make 10. Do you all agree?

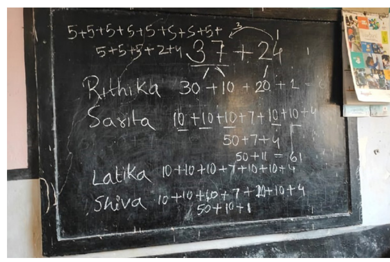
Students: Yes

Teacher: What else can we see?

Deepa: We can also say 10 – 7 = 3

Teacher: Do you agree with what Deepa said?

2) In Class 3, we worked on using different strategies to engage with double-digit addition. Students were asked to come forward and explain their way of dealing with the problem. The problem was 37 + 24.



Mention the name of the students along with their strategy

Figure 3

Teacher: Can you please explain how you approached the problem?

Rithika: I wrote 37 as 30 and 7, similarly 24 as 20 and 4. I added 30 and 20 which is 50, also added 7 and 4 which is 11. Then I added 50 + 11 which will be 61.

Sarita: I thought of 37 and 10,10,10 and 7. 24 as 10, 10 and 4. Later I added all the 10s and it will be 50. 7 and 4 are 11, so 50 and 11 are 61.

Latika used the same strategy as Sarita.

Teacher: What do you think is the most efficient way of approaching the problem?

Rahul: I think 10+10+10+…. Used by Latika and Sarita is an efficient one.

Teacher: Okay, so Rahul thinks using 'breaking into 10s' and adding is efficient. What do others think?

Sarita: I feel Rithika's method is more efficient.

Teacher: Why do you think so?

Sarita: Because she directly added 30 and 20, and further added 7 and 4 which is more efficient than adding 10 each time.

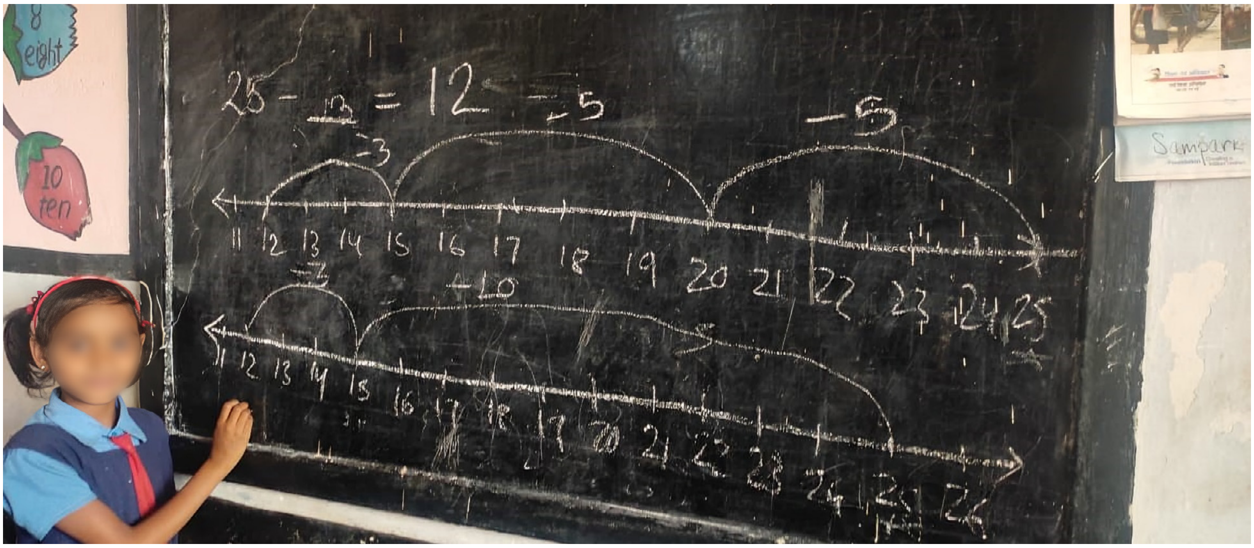
Suggested questions

Who has a different solution?

Have you thought of another way this could be done?

Whose method do you find more efficient?

The best part of this engagement was students started exploring and figuring out different methods to approach a problem, they provided reasoning for the problems instead of directly writing the answer which was previously valued the most in their classroom. They listened patiently to their classmates and the way they approached the problem. Even for missing number problems students were able to think instead of directly doing the problems mechanically. The below picture shows two methods used by a student for the same problem of missing addend.



Some points to remember while implementing classroom discourse:

1. It takes time to establish a mathematical discourse-rich classroom but it's worth all the effort. As students find their voice in the learning process, it makes the learning more meaningful and joyful.
2. One needs to identify and support the one not participating in the discourse as students might find the process overwhelming and need enough encouragement and support during the initial stage.
3. Assessing individual students while setting up such classroom culture could be difficult at the beginning but will get easier with proper planning.

Conclusion:

Mathematical discourse is a powerful tool to bring the required shift from a top-down mathematical learning process to a meaningful mathematical learning process. Fostering mathematical discourse in the classroom is essential for transforming mathematical learning from a passive to an active process in which students take ownership of their learning. By creating environments where students feel valued, supported, and empowered to communicate their mathematical ideas, teachers can nurture confident, rational thinkers and listeners. Despite challenges, the benefits of mathematical discourse outweigh the efforts involved, leading to deeper engagement, richer understanding, and a more inclusive learning community.

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ClassRoom

Abstract to Life: A Geometry Experience

ARUNI JOSEPH

This article shares my experiences of teaching in a government school in rural Rajasthan (Oct-Nov 2022) as part of the field practice of my M.A. Education programme at Azim Premji University. The experience in the field helped me understand the difficulties children face specifically in understanding mathematical concepts. I observed and attempted to address the challenges children faced in understanding angles in different orientations and in the measurement of angles. There was a scarcity of concrete materials which are very crucial for the teaching-learning of geometry and textbooks were used as the sole resource for teaching. A few pedagogical practices that were tried in order to address these concerns and which were found to be effective are discussed here.

How do children learn geometry?

Through geometry, children develop the power to imagine, discern elements that are not shown, visualise objects as dynamic and recognise facts and relationships that can be established (Johnston-Wilder, 2005). As per the theory of Pierre and Dina van Hiele, children progress through levels of geometric thinking (van Hiele, 1986). From the initial level of judging a figure by its appearance, children will identify the properties associated with the figures. After that, they draw relationships between properties and find one from the other. In the end, children will learn to reason in a formal mathematical way. Table 1 gives Van Hiele's levels of learning geometry.

Keywords: *Geometry, Angles, Conceptual Understanding, Mathematics TLM, Games.*

Table 1

Level	Name	Description	Example
1	Visual Level	Children recognise basic shapes by their appearance and name them.	A child at this level would identify a rectangle by its appearance.
2	Descriptive level	Children identify properties of basic shapes and can describe them.	A child would be able to identify a rectangle as a figure, with opposite sides equal and all angles right angles.
3	Abstract level	Children begin to establish relationships between the properties of shapes.	A square would be identified as a rectangle since it can be considered as a rectangle with additional attributes.
4	Formal deduction	Children construct original proofs which have a sequence of statements that logically justify a conclusion.	A child would be able to prove that the diagonal of a parallelogram divides it into two congruent triangles and justify the steps logically.
5	Rigour	Students establish theorems in different postulate systems and analyse/compare these systems.	The student will be able to prove or justify what are necessary and sufficient conditions to construct a quadrilateral.

My experience with teaching angles

I conducted over 25 sessions of 40 minutes duration each, for a class of 35 children in Grade 7. This was at the Mahatma Gandhi Govt School, Paldi-M in the Sirohi district of Rajasthan. Definition of angles, types of angles such as obtuse angles, measurement and drawing of angles using a protractor had already been taught to the children. The following were the observations made during the initial days:

- 25% of the children faced difficulty in understanding angles in different orientations.
- 70% of the children had challenges in angle measurements using a protractor.
- 5 children were found to be disinterested in doing paper and pen work in class.
- Children were not exposed to any teaching-learning materials for angles.

Pedagogical practices that were tried

To address the difficulties faced by the children, a few pedagogical practices were tried which included the use of concrete materials and the extension of the content in the textbook. These served as a vehicle to transact the topics of angle representation, angle measurement, angle pairs such as complementary and supplementary angles, and angle pairs formed when a transversal cut a pair of parallel lines.

a. Using Teaching Learning Materials (TLMs)

TLMs can be effective in helping children navigate the abstraction in angles. Learning how to represent angles mathematically is not straightforward for young children, even though angles occur everywhere in their daily life (Watson, 2013). A few materials tried in the class are given below, mostly made of recyclable materials like newspapers.

Paper Fans

Most of the time, our teaching focuses on visualising angles formed when two lines/ line segments intersect. There is evidence that children are more aware of angle in the context of movement (turn) than in other contexts and learn about the mathematics relatively easier in this context (Nunes, 2009).

Figure 1

Paper fans (Figure 1) were made using ice-cream sticks and newspapers and were used to show angle in the context of movement. Children did the activity in groups of 5 members. They were told to describe what they saw while using the fans. They formed different types of angles and described them. "*This is more than 90° and is an obtuse angle.*"

They used angles in different orientations unknowingly as they used the fans. A few questions were asked to assess their learning. For example, an obtuse angle was shown to the class. Children recognised it as an obtuse angle. Thereafter, the orientation of the angle was changed without changing the measurement and they were asked what type of angle that was. Suman, who used to be very quiet in class, said it was an obtuse angle. When asked why it was so, her reply was, "You did not change the positions of the sticks, you just turned it upside down. So, it is the same angle." The group activity encouraged engagement and discussion among children. Classroom tasks can be used to give children opportunities to explain, justify and be critical of their own and peers' explanations (Watson, 2013).

360° Protractor

Many children find it challenging to measure the angle in a given figure, especially when the figure is in an uncommon orientation. 360° handmade protractor (Figure 2) was used as a visual aid to form angles and measure them at the same time. This also gave the opportunity to explore angles greater than 180° which is a limitation of the conventional protractor. This material with two hands was made using chart papers and secured using a pin and a sponge piece. Children explored this material in groups of four members and were asked to make angles of different types and measures.

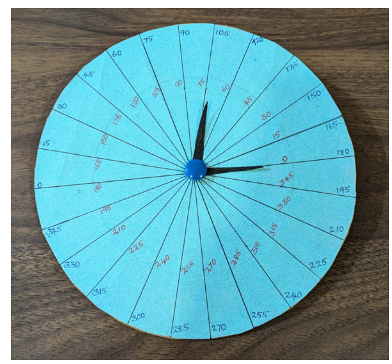


Figure 2

They described what they made. "*This is 120° and is an obtuse angle*", "*When we make an obtuse or an acute angle, we also get a reflex angle at the other side*". When asked to show 30° angles in three different ways, some of the responses given are shown in Figures 3, 4 and 5. After these activities, there was a noticeable improvement in measuring and drawing angles using protractors.

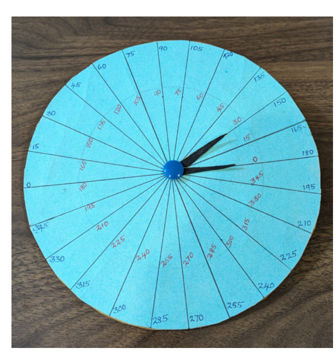


Figure 3

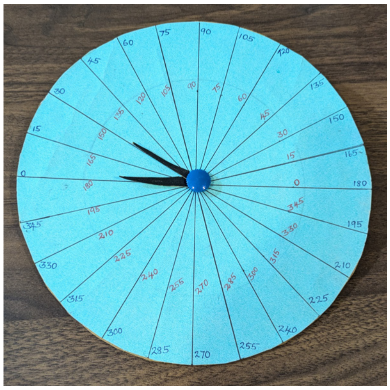


Figure 4

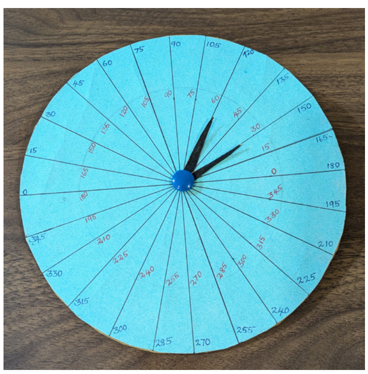


Figure 5

Coloured straws

In groups, children were asked to explore and identify angles around the transversal cutting two lines. Children concluded that there were eight angles in such a scenario.

Figure 6

Model of transversal cutting two parallel lines

The model in Figure 7 was used to find relations between angles formed when a transversal cuts pair of parallel lines. For example, what can we say about the measures of ∠AXP and ∠CYX? What about ∠AXY and ∠DYX? Once children articulated the relationship, we moved towards learning their universally accepted names - corresponding angles and alternate interior angles respectively.

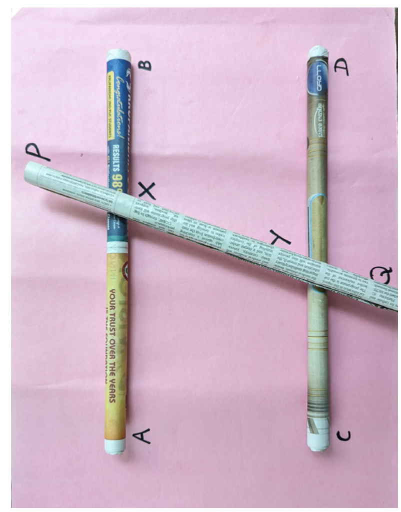


Figure 7

The model's orientation was changed (Figure 8) to check if children identified the pair of angles they worked with earlier, and their relationship. Children could make out that the change in orientation did not change the measure of the angles or the relationship between specific pairs of angles.

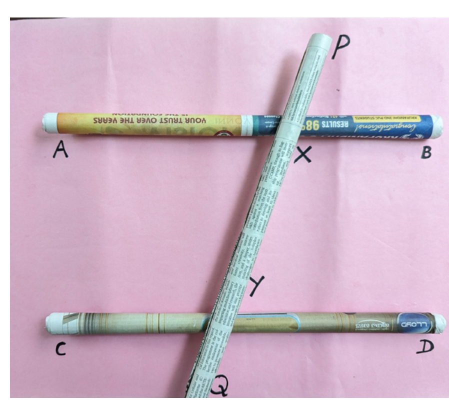


Figure 8

b. Extension of textbooks

A few activities and guiding questions were prepared using NCERT textbooks.

Gamification of textbook contents

One such activity was on complementary and supplementary angles. Children were divided into 2 groups. Pictures of different measures were given to the children of one group (Figures 9, 10, 11, 12). They had to find the complement or supplement of each angle which was with the children of the other group.

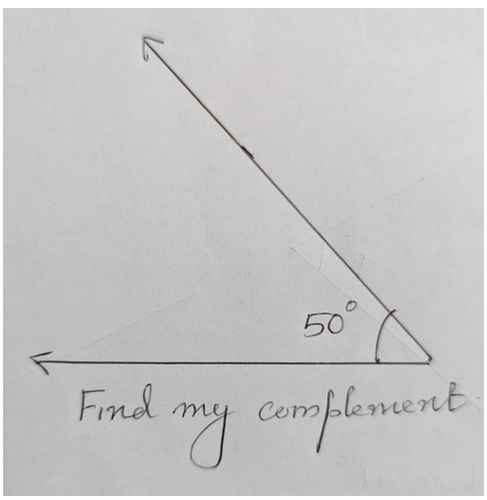


Figure 9

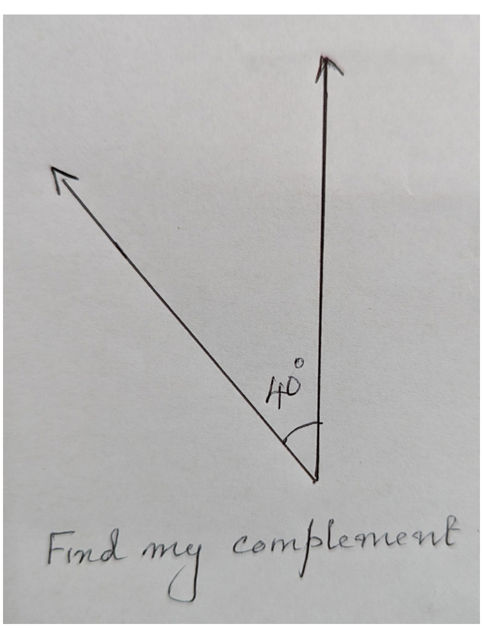


Figure 10

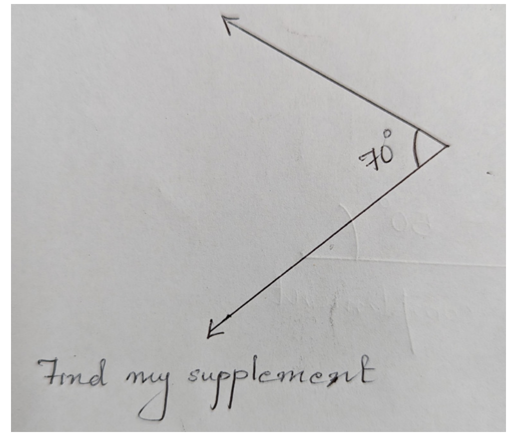


Figure 11

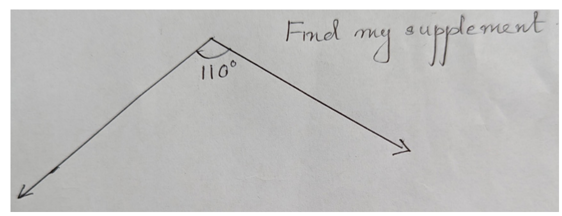


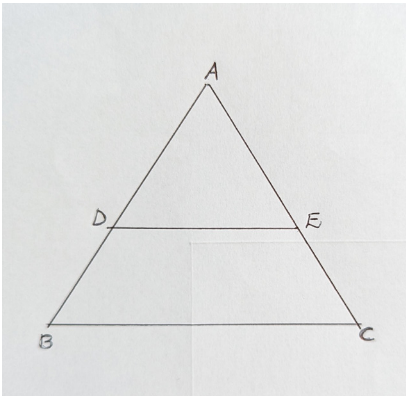
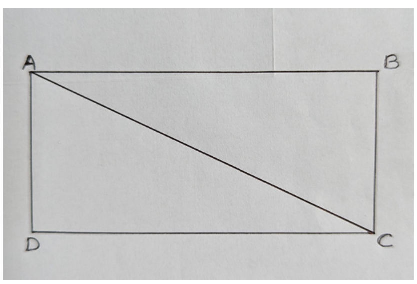
Figure 12

Children were engrossed in the activity by calculating, estimating and verifying with the pictures. This activity was found to be beneficial for children who were not interested in paper and pen work. Post the activity, children were asked to describe what they saw in the angle pairs. This was to understand whether they saw the sum of the angle pairs as 180° and 90°, and hence could differentiate between supplementary and complementary angles respectively.

Guiding questions

Guiding questions were used to enable children to apply higher-order cognitive skills and to prepare them for formal mathematical thinking in secondary school. The angles that came alive would be more meaningful when they saw their application in a different context. Table 2 has a few such scenarios tried in 7^{th} grade and based on content from the 9^{th} grade NCERT textbook. To do these, students would need to know representation and types of angles, angle pairs such as supplementary and complementary angles, as well as angle pairs formed when a transversal cuts a pair of parallel lines.

Table 2

<table>
<tr><th>Excerpts from textbooks of higher classes</th><th>Guiding questions</th></tr>
<tr><td>
</td><td>1. Identify the transversals
2. Identify the corresponding angles.</td></tr>
<tr><td>
</td><td>1. Identify the transversals.
2. Identify alternate interior angles.
3. Identify co-interior angles.

Remarks
Most of the children identified AC as a transversal. Kamal even remarked that the sides of the rectangle are also transversals. This led to a discussion where children put across their points, debated and concluded, which prompted me to ask the third question. There is also a possibility to identify complementary angles in this figure which could be used for further discussion.</td></tr>
</table>

Children would get the opportunity to use higher-order skills such as applying and evaluating from activities like these. When children are asked to find solutions in groups, it becomes an exploratory activity, making them think, propose, make mistakes and learn. A geometry teacher can thus accelerate children's progression in the Van-Hiele model. Classroom tasks should provide opportunities for students to develop problem-solving skills and engage in problem-posing (Watson, 2013).

Key findings

- Manipulatives such as paper fans and 360° protractors were found to be effective in giving children exposure to angles in the context of movement, angles in multiple orientations and angles as relational measures. There was a noticeable improvement in the measurement and construction of angles using protractors after this.
- 360° protractors can be used in overcoming the limitations of a conventional protractor where angles more than 180° cannot be shown. If made using transparent sheets this can also be used for measuring and drawing angles.
- Introducing games and guiding questions encouraged inquisitiveness in children, thus making learning joyful and participative.
- Giving children opportunities to explore using concrete materials and effective textbook usage accelerated their progression in van Hiele's model of learning geometry.

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Aruni Joseph started her career as an IT professional. After spending a decade in the industry, she moved on to pursue her interest in teaching Mathematics by taking after-school tuitions. She completed her M.A. in Education from Azim Premji University in 2023. After that she worked with the Gubbachi Learning Community in Bengaluru, teaching Mathematics to the out-of-school children of migrant labourers. Currently she works in Prakriya Green Wisdom School, Bengaluru. Aruni may be contacted at arunipradeep@gmail.com

A Journey from Euclidean to Non-Euclidean Geometry

Euclid's postulates, Elements' true signature,
Akin to nature's basic elements, a divine scripture.
Euclid's five, building blocks to Geometry,
Nature's five mirror our being's symmetry.

Two points uniquely determine a straight line,
Extend indefinitely, a terminated line's design.
Circles constructed, varied in size and location,
Congruent right angles, Euclidean geometry's foundation.

The fifth, the Parallel postulate, sparked a mathematical divergence,
Paving a way to non-Euclidean geometry's emergence.
A traditional shift, where parallels take an unusual stance,
In hyperbolic and elliptic realms, mathematical refinement enhance.

Euclidean geometry initiated through Euclid's sage,
Riemann's elliptic geometry, an elegant grace.
Lobachevsky, Bolyai, and Gauss in hyperbolic fervour,
A tribute to minds with questioning valour.

Composed by:
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ClassRoom

Why 6? Connecting Tetrahedral Numbers to a Tetrahedron

BRAD UY AND JAMES METZ

Triangular numbers have their name because a triangular number of circles can be arranged in the *form* of an equilateral triangle, as shown in Figure 1.

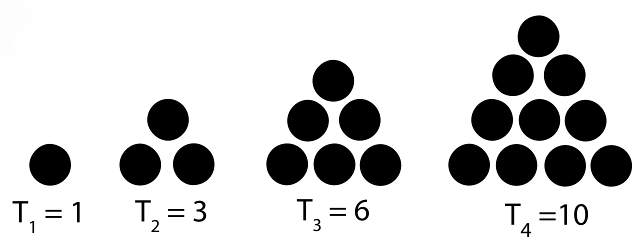
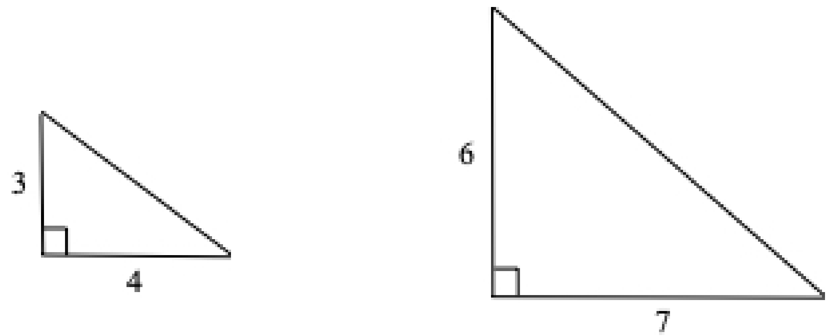


Figure 1. Triangular numbers represented by circles

The first triangular numbers are 1, 3, 6, 10, 15, 21, … At step number 2 we add 2 more circles, at step number 3 we add 3 more circles, so in general at step number n we add n more circles. This is how we generate triangular numbers, but if we wanted to know the tenth triangular number we need to know the ninth triangular number. We need a general formula to find the n^{th} triangular number without knowing the previous one. The general formula for the n^{th} triangular number is $T_n = n(n + 1)/2$. (For example, $T_{10} = 10(9)/2 = 45$.) This formula has a nice geometric representation as the area of a right triangle with legs n and $n + 1$. See Figure 2. Since $n(n + 1)/2 = (n^2 + n)/2$, the n^{th} triangular number is also the average of n and n^2.

Keywords: *Special numbers, triangular numbers, tetrahedral numbers, geometry, visualisation*



Area = 3•4/2 = 6 = T_3 Area = 6•7/2 = 21 = T_6

Figure 2. Right triangles whose legs are consecutive positive integers

While triangular numbers can be represented as circles arranged in the *form* of a triangle, tetrahedral numbers can be seen as a *form* of a tetrahedron, a pyramid with a triangular base, as in Figure 3. We create tetrahedral numbers by stacking spheres, with a triangular number of spheres in each layer. Figure 3 shows the sum of the first three triangular numbers, 1 + 3 + 6 = 10, the 3rd tetrahedral number.

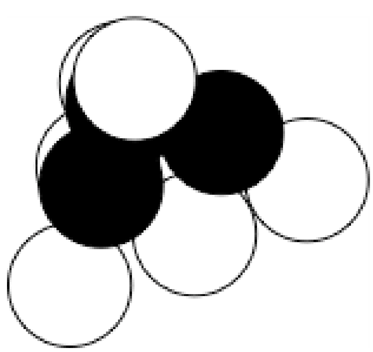


Figure 3. Stacked spheres showing the 3rd tetrahedral number, 10

The first tetrahedral numbers are 1, 4, 10, 20, 35, 56, ... While the model does provide motivation for naming the total number of spheres as tetrahedral numbers and it is a nice visual representation of the relationship between triangular numbers and tetrahedral numbers, it does not help us find the n^{th} tetrahedral number unless we know the one before it. As with triangular numbers, we want a general formula to the n^{th} tetrahedral number. This formula for the n^{th} tetrahedral number is $T_n = n(n+1)(n+2)/6$. (For example, $T_5 = 5(6)(7)/6 = 35$.) Wouldn't it be nice if we could picture this expression geometrically using a *tetrahedron*?

The numerator of the expression that generates tetrahedral numbers is the product of three consecutive positive integers which is conveniently also the volume of a rectangular solid whose dimensions are three consecutive positive integers. Thus, 1/6 of the volume of such a rectangular solid is always a tetrahedral number. For example, a rectangular solid with dimensions 3 × 4 × 5 has a volume of 60, so the third tetrahedral number is 60/6 = 10. Figure 4 shows 6 sets of 10 cubes, each with a unique colour for each set. These 6 sets fit together to form a 3 × 4 × 5 rectangular solid. It is not necessary that cubes in a set be joined or even that cubes of the same colour stay together (the cubes may be anywhere in the solid), but this configuration nicely reveals the triangular numbers 1, 3 and 6 used to generate the tetrahedral number 10. An alternative model is shown in Figure 5, a model suggested by Swati Sircar of Math Space, Azim Premji University.

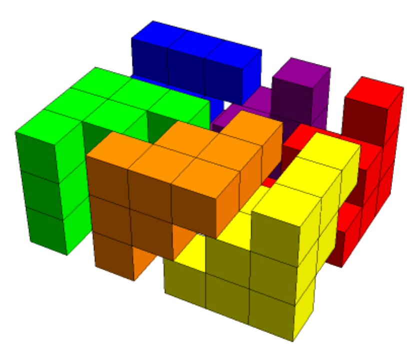
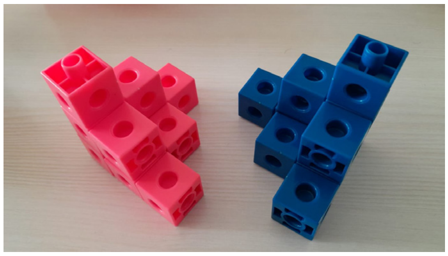
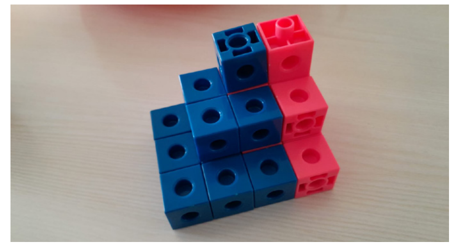


Figure 4. Each of the 6 identical pieces has 10 cubes.

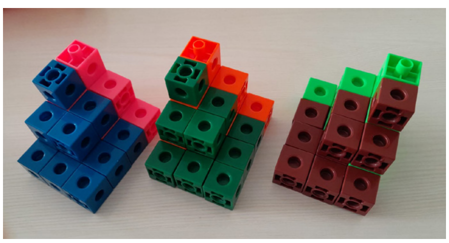
Step 1. Each of the two sets has 6 + 3 + 1 = 10 blocks for a total of 20 blocks.



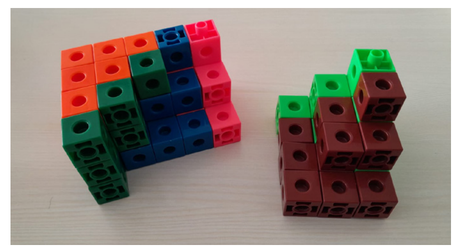
Step 2. The bottom 6 blocks in each set join to make a 3 by 4 rectangle; the middle 3 join to make a 2 by 3 rectangle; the top block in each set join to make a 1 by 2 rectangle.



Step 3. Three identical sets of 20 blocks.



Step 4. Two sets are joined.



Step 5. The three sets are joined together to form a 3 × 4 × 5 rectangular solid.

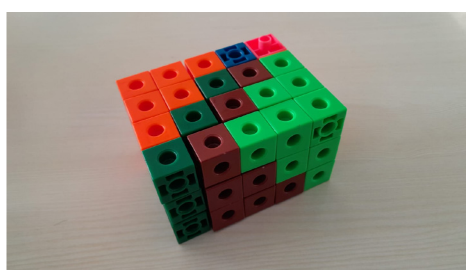


Figure 5. Swati's Construction

This model starts with the formula and fits the model to the formula, so this method does not answer the question, "Why 6?"

We now turn our attention to the possibility of picturing this expression geometrically using a *tetrahedron*. We must be clear that this is not an effort to develop the formula for the n^{th} tetrahedral number, but rather to show a way to see the expression in a tetrahedron, and notably to account for the number 6 in the expression. We again observe that the n^{th} tetrahedral number is the sum of the first n triangular numbers. For example, the sum of the first five triangular numbers, 1 + 3 + 6 + 10 + 15, is 35, the fifth tetrahedral number.

The formula for the volume of a tetrahedron is $Bh/3$, where B is the area of the base and h is the height. If a tetrahedron has a right triangular base with legs n and $n + 1$, the area of the base is $n(n + 1)/2$, a triangular number. If the height of the tetrahedron is $n + 2$, then the volume of the tetrahedron, is $[(n(n + 1)/2)(n + 2)]/3)$ or $n(n+1)(n+2)/6$. See Figure 6. Thus, when asked to determine the n^{th} tetrahedral number, we simply compute the volume of a tetrahedron. For example, the fifth tetrahedral number is $Bh/3 = [[(5)(6)/2])7]/3 = 35$.

We can now answer, "Why 6?" We have a divisor of 2 from the formula for the area of the triangular base and a divisor of 3 from the formula for the volume of a tetrahedron and hence the divisor is 6.

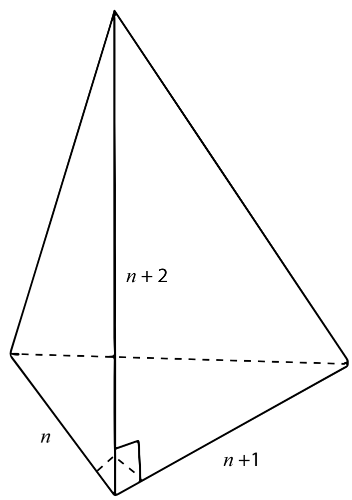


Figure 6. The volume of this tetrahedron is the n^{th} tetrahedral number.

We conclude with a final thought. A popular song in some parts of the world is "The Twelve Days of Christmas," whose first three verses are:

On the first day of Christmas
My true love gave to me
A partridge in a pear tree.

On the second day of Christmas
My true love gave to me
Two turtle doves,
And a partridge in a pear tree.

On the third day of Christmas
My true love gave to me
Three French hens,
Two turtle doves
And a partridge in a pear tree.

This pattern continues for a total of twelve days. The number of gifts received each day is a triangular number, so the total number of gifts is the twelfth tetrahedral number, the sum of the first twelve triangular numbers, 364. The song and lyrics are widely available and provide a nice introduction to the problem.

Reference

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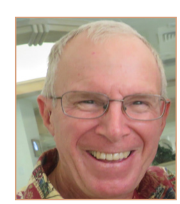

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ClassRoom

Multiplication: A Better Algorithm

MATH SPACE

How do we multiply two numbers like 84 × 67? At the first step we find 84 × 7. This involves a 'carryover', or regrouping, i.e., 84 × 7 = (4 + 80) × 7 = 4 × 7 + 80 × 7. So,

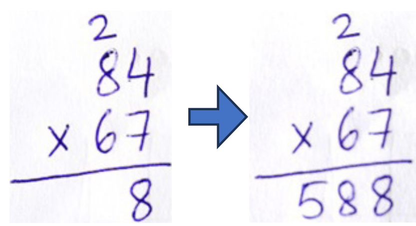

Figure 1

- We start with 4 × 7 = 28
- And put down 8
- Remember that we need to add 2 to the next product
- Find 80 × 7, or rather 8 × 7 = 56
- And add 2 to this, i.e., 56 + 2 = 58
- Complete the product 84 × 7 = 588

And this is just step 1 (Figure 1)!

Note the complexity! When we add, we deal with only one operation and all digits are added. Here on the other hand, to multiply by a 2-digit number, one has to multiply 8 and 7 and then add 2. In other words, there are two operations in this step – multiplication and addition. Therefore, the order of these two operations matters since (7 × 8) + 2 ≠ 7 × (8 + 2). But it is quite natural for a learner to get confused on whether to multiply (by 7) first and then add, or the other way. Also, unlike addition, the 'carry' is not usually written down. So, there are more things to keep in mind.

Keywords: *Multiplication, algorithm, two-digit, lattice, conceptual understanding*

A similar set of steps is repeated for 84 × 60, or 84 × 6 = 504. This also requires 'carry' and since writing multiple such digits can be confusing, none are usually written. But that hardly helps a learner at the beginning.

Moreover, this is often written with a blank or '×' at the unit's place (Figure 2). Then we add up 588 and 504×. No one is taught to add 8 + × but is expected to do so here. It is surprising how this continues in classrooms still, at least 20-25 years after textbooks have changed. Since we are multiplying by a multiple of 10, i.e., 60 in this case, why shouldn't we write the partial product of 84 × 60 as 5040? We hope that teachers stop using the '×' out of sheer inertia to change and make their pedagogy more meaningful.

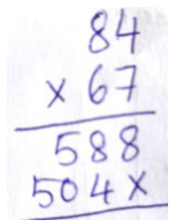

Figure 2

Is there another way to avoid such issues? Fortunately, there is one. And it can be used for any multiplication with two whole numbers, no matter how large each is.

The standard method, illustrated above uses distributive property only once, viz.,

$84 \times 67 = 84 \times (7 + 60) = 84 \times 7 + 84 \times 60$

which results in the two partial products 588 and 5040 that gets added at the end.

However, we can use the distributive property twice, i.e., for both numbers and get four partial products:

$84 \times 67 = (80 + 4) \times (60 + 7)$

$= (80 \times 60) + (4 \times 60) + (80 \times 7) + (4 \times 7)$

This can be tabulated very nicely in a 2-way table (Figure 3). And then the partial products can be added up. Note that this is not radically different from the earlier method. The column-wise sums are the same partial products which we got earlier.

×	60	7
80	4800	560
4	240	28

Figure 3

For a 3-digit × 3-digit product, say 379 × 825, this would become Figure 4. However, as the number of digits increases, writing all the zeros may become cumbersome. But there is a way out as well!

×	800	20	5
300	240000	6000	1500
70	56000	1400	350
9	7200	180	45

Figure 4

We can use a smaller lattice, e.g., Figure 5 for 84 × 67. But to keep track of the zeros in the partial products (as shown in the big lattice in Figure 3), it helps to diagonalise each cell (Figure 6).

×	6 tens	7 units
8 tens	48 hundreds	56 tens
4 units	24 tens	28 units

Figure 5

×	6 tens	7 units
8 tens	4 / 8	5 / 6
4 units	2 / 4	2 / 8

Figure 6

Note how the digits in each diagonal (bottom-left to top-right) correspond to the digits in the sum of partial products (right to left) as shown in Figure 7. This is explained if we use arrow cards to make the big lattice and colour the small one accordingly (Figure 8).

28
240
560
+ 4800
5628

Figure 7

So, we could sum up diagonally as well (Figure 9).

Therefore, if we crystalize this method, it boils down to:

- Create a 2-way table with the numbers to be multiplied
- Draw diagonals in each cell
- Multiply digit-by-digit to fill each cell
- Add along each diagonal group of digits starting from the bottom-right

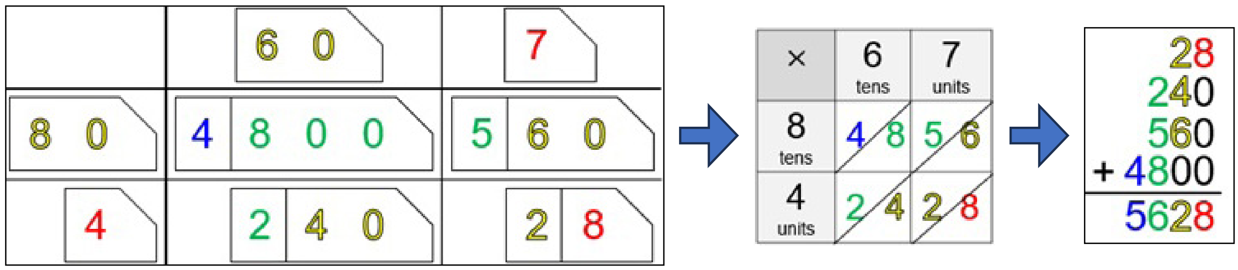

Figure 8

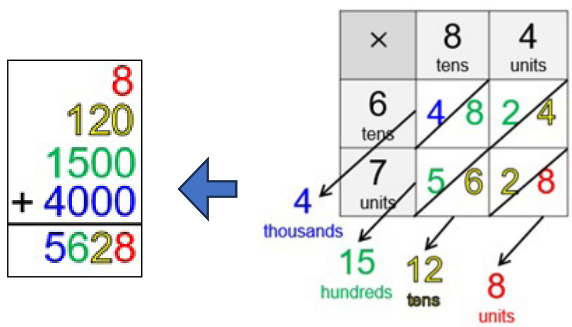

Figure 9

Let us illustrate this for 379 × 825:

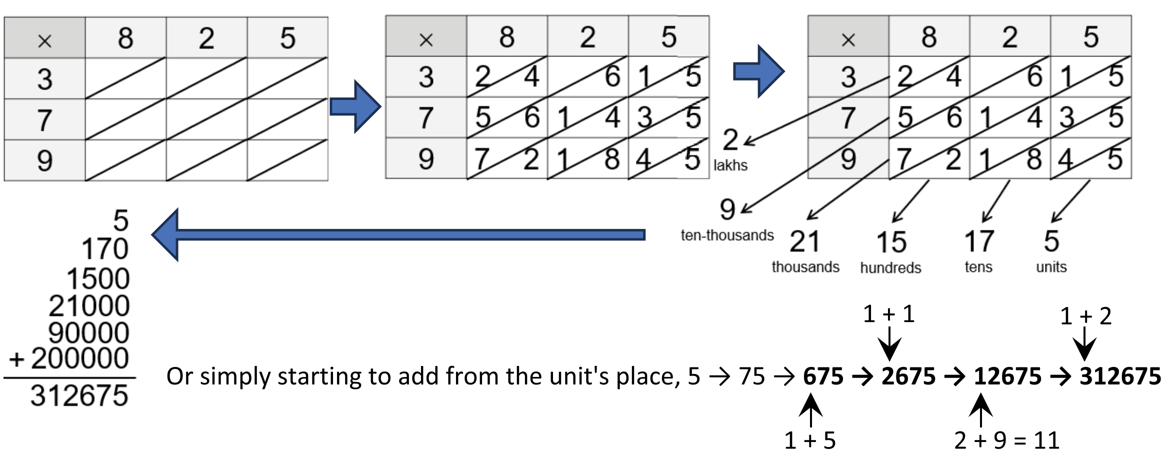

Figure 10

Note that in this method, when we multiply, it is just digit-by-digit multiplication with no addition, and therefore no 'carry' or regrouping. And once all multiplication is done, we add up. So, this is a written process like the previous one. But without the juggling of multiplication and addition. And no chance of any "×" instead of 0 (as placeholder).

However, pedagogically speaking, one should always start with the big lattice and then transition to the small one using arrow cards. It helps a lot to start with some 2D base-10 blocks aka flats-longs-units (FLU), reviewed in the Mar 2024 issue of this magazine [3], for up to 2-digit × 2-digit products. But for bigger ones it helps to draw lines and count the number of intersections. The lines should be colour-coded to separate ones from tens, etc. Number of intersections of each type can be used to fill the 2-way table with diagonals. We illustrate this for 379 × 825 (Figure 11). [Drawing intersecting line for multiplication is popularly known as lattice method.]

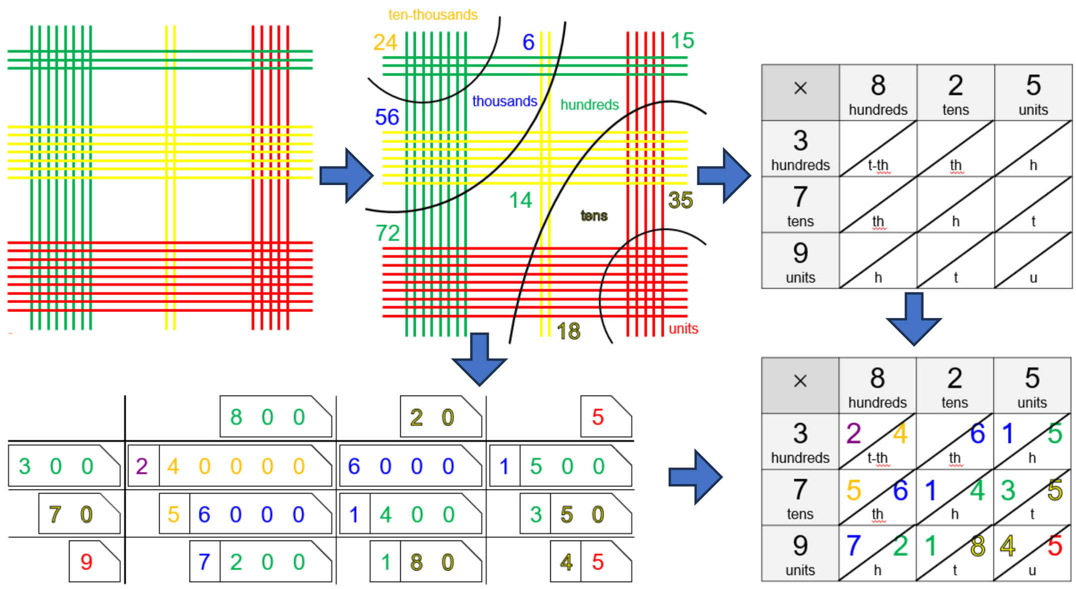

Figure 11

The need to multiply big numbers by hand has certainly reduced in real life thanks to the prevalence of technology. However, at school and therefore in board exams, learners have to multiply multi-digit numbers by hand. Various problems in percentages and mensuration may require computing such products. We hope that this method provides a simpler way to do that compared to the standard process.

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MATH SPACE is a mathematics laboratory at Azim Premji University that caters to schools, teachers, parents, children, NGOs working in school education and teacher educators. It explores various teaching-learning materials for mathematics [mat(h)erials] their scope as well as the possibility of low-cost versions that can be made from waste. It tries to address both ends of the spectrum, those who fear or even hate mathematics as well as those who love engaging with it. It is a space where ideas generate and evolve thanks to interactions with many people. Math Space can be reached at mathspace@apu.edu.in

The Art of Guesstimation - Part 2 Using Fermi Problems in a Classroom Setting

MOHAN R

In the first part of this article published in the March 2024 issue, we learned about *Fermi Problems* and looked at some examples. Fermi problems challenge us to make educated guesses to solve complex questions with limited information. These guesses are based on intuition and common sense, which makes the problems engaging and relatable. Often connected to real-life scenarios, they allow us to apply basic mathematical concepts in practical ways. This makes them a valuable teaching tool in the classroom. In this second part, we will explore more examples of Fermi problems suitable for classroom use and discuss effective ways to present them to engage students.

Planning the classroom activity

As we saw in the first part, solving Fermi problems involves various strategies, assumptions, and estimations. In the classroom, it's effective to treat these problems as discussion-based activities. This approach requires careful planning and selection of problems that are appropriate for the learner's age, local context, and mathematical preparedness. In this section, we will explore a sample activity along with a few suggestions.

Every good activity starts with an engaging introduction. To kick off the discussion, the teacher can pose an intriguing, amusing, or surprising question that requires students to estimate an answer. For instance, consider these prompts:

- 'For how many seconds are your eyes closed in a year?'
- 'How long would it be if you laid all the strands of hair on your head end to end?'

Keywords: Fermi Method, Estimation, Creative Problem-solving, Reasoning

The key to a good introduction is simplicity and lack of ambiguity, ensuring every student can immediately engage by making and justifying their own guesses. With the provided examples, students might wonder whether blinking counts, how long they sleep each day, the average duration of sleep; the average length of a hair strand, or how many strands of hair they have on their heads. Such questions might further stimulate instant experiments!

Next, the teacher can guide students in making informed guesses and estimations. This might involve introducing the concept of Fermi problems with some standard examples - such as those discussed in the first part of the article -and working through their solutions. Discussing Enrico Fermi's story briefly could also be enlightening. The class could then be divided into small groups of two or three students each, with each group tackling a specific problem.

The teacher continues the activity by reading out the problem and facilitating a discussion on what the question means and how students might interpret it. It's useful to discuss that a range of answers might be acceptable. The students could make their immediate guesses, which they need to record. Then they could be asked to come up with a strategy to find a better estimation. This could involve a series of smaller estimation problems, each requiring reasoning and calculations based on everyday experiences and common sense. Observing how different groups approach the problem and the assumptions they make at each step can be a valuable learning experience. It is helpful if the problems are chosen so that a few steps can be experimentally verified (for example, measuring).

The teacher can help the groups define variables and come up with formulas to solve their steps better. She could then ask groups for their estimates to show the range of answers possible, and also periodically share the various approaches with the class. Students need to understand that we are seeking an order of magnitude estimate, not an exact number, which in the context of Fermi problems, is not possible to calculate.

Students should identify numbers that are likely to be too small or too large to be the answer. Next, they should make guesses that they think are close to the correct answer. During the final discussion, students should share their assumptions and estimates. They can compare their guesses to the answers they found and the data they gathered/justification of their answers to be made. It is useful to discuss whether the answer might change if the experiment were repeated, and what factors might contribute to different answers. Finally, the class could discuss open questions that generate more questions to investigate.

A sample activity

Let's explore an example of how an ideal classroom activity might unfold. Having already introduced the Fermi problems and divided the students into groups, the teacher now engages them in the problem-solving activity. The scenario is set in a 7th grade classroom.

Teacher: Alright, now that we know what Fermi problems are and how to approach them, here is a question for *you* to solve: How many people would be needed to surround the state of Madhya Pradesh if they all held hands?

[Students murmur, some look intrigued, others puzzled.]

Teacher: Before we start guessing, let's try to take up an easy problem. Suppose all of us stand side-by-side holding each other's hand and form a circle, how large will the circle be?

One way of answering this question is to find the length of your arms while they are stretched to the sides, making a 'T' shape. Then we can multiply the length by the number of children in the classroom. This would give us the *perimeter* of the circle. Do you think there will be any problems with this approach?

Student 1: Yes, some students have longer hands. So, the answer will depend on whose arm length is measured.

Teacher: Good observation! I have brought a measuring tape. Let us quickly measure the lengths of five children and check how much the lengths vary.

[The teacher measures and records the lengths on the board.]

Teacher: As you can see, we are measuring the lengths in *metres*. Depending on whose measurement is taken, the final answer changes. So, 0.9 metres multiplied by 25 (persons) =22.5 metres, and 1.1 metres multiplied by 25 = 27.5 metres. Because we are finally calculating a very large number, let us not worry so much about the decimals or one's place, but only the tens place. So roughly speaking, the length of your arms would be around 1 metre, and that multiplied by the total number of students would come between 20-30 metres, right?

Now, let us try to answer a slightly related, but different question: Suppose we are inside our rectangular classroom, standing close to the walls with our hands held together. How many children would be required to cover the room?

Student 2: To find that we should measure the perimeter of the classroom.

Teacher: Exactly! [Assuming the classroom is in a rectangle shape, the teacher measures the length and breadth of the room and asks the students to compute the perimeter.] Now that we know the perimeter of the room in metres, and we assume that each student's arms-length is 1 metre, so we can mention the unit of the length in terms of number os persons. This means we can say 25 persons instead of 25 meters.

Can you guess how many students it would take to surround the school building?

Figure 1: [Students make guesses without measuring the lengths.]

Teacher: Now, let us make our question a little interesting. How many people would it take to surround the school ground? What about the school compound? What about the village?

[Students make guesses and share their reasoning.]

Teacher: Do you think there will be any issue when we surround the village? Is the ground surface even?

Student 3: No, there is a lake nearby and some places are in higher elevation compared to others.

Teacher: Yes, this is a problem we would have to address by again assuming that the ground level is the same around the village. Also, we typically get the measurements in kilometres when we measure large distances, so we have to convert them to metres.

Now, let's come back to our original problem, here is the map of Madhya Pradesh. As you can see the boundary of the state is not even like a circle, so we need to make a few assumptions and find the perimeter of the state. Since most of the people are adults, whose arms-length is bigger than that of children, let us also assume that the arms-length vary from 1 metre to 1.5 metres. With these additional assumptions, let us now try to find the number of people it would take to surround the state.

[The activity continues.]

A few tips in order:

- This activity is also a great opportunity for students to practice their written and verbal communication skills. Students often enjoy sharing their findings with other students at the end of the activity.
- Solving Fermi problems often involves converting units and hence introducing *dimensional analysis* as a bookkeeping tool can be useful.
- Providing students with simple calculators can also expedite calculations.

In summary, it's helpful to guide students through a structured process instead of just making random guesses. Here's a revision of the step-by-step process:

1. Start with the question and make sure everyone understands it.
2. Make a rough guess without doing any calculations.
3. Make an educated guess with reasoning and calculations based on everyday experiences and estimates.
4. Define variables and create formulas to solve the problem.
5. Conduct experiments, measure things, and find information to improve estimates and figure out the smallest, largest, and most likely values for the answer.
6. Summarise the conclusions, note possible errors, and interesting facts learned, and suggest directions for future investigations.

Below, we provide a set of sample Fermi problems sourced from the internet, categorized by age range. Teachers can adapt these problems as needed to fit their classroom settings and local contexts.

Fermi problems for the age range of 4-8

Young students should have concrete Fermi problems that they can understand and complete. They should choose two occasions to pause and check their guesses - one early on and one around halfway through. After completing the task, students should use pictures, numbers or equations, and words or sentences to show what they did and what they learned.

Here are a few suggestions for Fermi questions for this age level.

1. How many blocks do we need to stack to reach your height?
2. How many stickers would cover this notebook?

3. How many punctuation marks (or letter a's, etc) are in this book?
4. How many rangolis would cover this hall floor?
5. How many bananas do we eat in a week?
6. How many jumps would you need to travel across the room?
7. How many beads would it take to make a bracelet that fits perfectly on my wrist?
8. How many tablespoons of water would fill this container?
9. How many cars pass by the school bus stop in a minute?
10. How many times do you blink in a minute?

Fermi problems for the age range 9-11

Students in this age range usually understand Fermi questions better if they can touch and interact with the things involved. They should at least start doing what the question asks physically, though it may not be practical to complete it.

Here are a few ideas for Fermi questions for this age level.

1. How many one rupee coins are needed to equal your height, the height of the school, the tallest building in the world, Mount Everest, and outer space?
2. How many grapes are needed to equal your weight, the weight of a car, the weight of the school building, of the Earth?
3. How many times would a one rupee coin roll to travel down the hall?
4. How many laddus are required to fill this room?
5. How many people would be needed to surround the state of Madhya Pradesh if they held hands?
6. If you prepared a tank with all of the air you need to breathe in one day, how large would it be?
7. How many blades of grass are in our school ground?
8. How many seconds does a student sleep per week?
9. What would it take to make a mural that runs the length of our school wall? (How much time, cost, people, material).
10. How much water is wasted by a leaking water tap in one day?

Helping students solve a Fermi problem

A good Fermi problem makes students ask more meaningful mathematical questions and solve them. For instance, if we ask about the cost of housing people in a rescue camp for a week, students would need to think about questions like how to source food, clothes, and other essential items. They'd also consider the cost and time for getting, storing, and transporting these things, which involves concepts learned from various units including geometry, ratio proportions, etc.

The internet is a great resource for finding many Fermi problems that can be modified for local contexts. It is also useful to remind the students of instances from local contexts where Fermi-type estimates are used. For instance, a farmer would like to estimate the yield of mangoes on his farm, a fisherman would like to estimate the yield of fish he would catch in a particular season, or a vendor would like to estimate the number of buyers during a festival season etc. It is also highly recommended that the students come up with their own Fermi problems, find strategies to solve them and share them in the classroom.

Conclusion

One of the aims listed under mathematics education in the NCFSE-2023 reads that

> "Developing an intuition for what should or should not be true in Mathematics is often just as important as the more formal 'paper-pencil' doing of Mathematics. Focusing on the common themes and patterns of reasoning across mathematical areas, guessing correct answers (in terms of, e.g., 'order of magnitude') before working out precise answers, and engaging in informal argumentation before carrying out rigorous proofs are all effective ways of developing such mathematical intuition in students."

Fermi problems provide a great opportunity for the students to not only develop intuition in terms of estimation skills but also provide them an opportunity to hone their mathematical communication skills.

Also, NCFSE-2023 lists, among others, one of the current challenges with respect to mathematics learning that

- "Mathematics learning has traditionally been more 'robotic' and 'procedural' rather than creative and aesthetic. This is a misrepresentation of the nature of Mathematics and must be addressed in the school curriculum.
- Very often, the content presented in textbooks to illustrate mathematical concepts is far removed from the contextual realities of the learners. Young students find some mathematical concepts easier to absorb when they are directly connected to their experiences. Textbooks, classroom activities, and examples should aim to be motivated by and related to students' lives whenever possible.
- There has also been a mistaken and exclusive emphasis on symbolic language and formalism in Mathematics teaching and learning, rather than on the informal argumentation and development of mathematical intuition that is so important for mathematical discovery."

Again, Fermi problems provide opportunities for the mathematics teachers to pose meaningful mathematical questions creatively that have direct relevance to the students' everyday lives. They provide another way to look at mathematics and explore. We urge you to try a few Fermi problems in your classroom and write back to us sharing your experiences.

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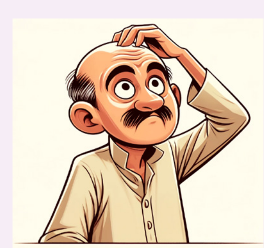
Contributed by Vijay Ravikumar, Azim Premji University

Here is a fun Fermi problem: Consider every person living on the earth. Suppose each person p has H_p number of hairs on their head. What is the product of all such H_p?



Answer: Zero. All that it takes is one bald person to make the calculation easy!

Special Year

JAMES METZ AND
BRAD UY

Our friend Laurie was born in 1962, and this year she will turn 62 years old. We found this interesting. We told her this is her 'Special Year.' We then wondered how individuals can determine their Special Year, the year in which the last two digits of their birth year match their age, given their birth year, b. The interested reader may explore this before continuing reading.

For Laurie, one can easily see that in her Special Year, her age is the difference between her birth year and 1900. We simply add that difference (the last 2 digits of her birth year) to her birth year to find her Special Year. Thus, we can know quickly that since Laurie was born in 1962 her Special Year is 1962 + 62 or 2024 and that she will be 62 years old then.

In general, consider a person born in year b between 1900 and 1999, inclusive. A person a years old in their Special Year, S, has the last two digits of their birth year $b - 1900 = a$, and since $S = b + a$, by substitution, $S = b + (b - 1900)$. A person born in 1949 was 49 in 1998, so 1998 is their Special Year. For a person born between 1900 and 1999, who is a years old, their Special Year S can be calculated as $1900 + 2a$. Since 1900 and $2a$ and are both even, the Special Year S, must always be an even-numbered year. (See Figure 1.)

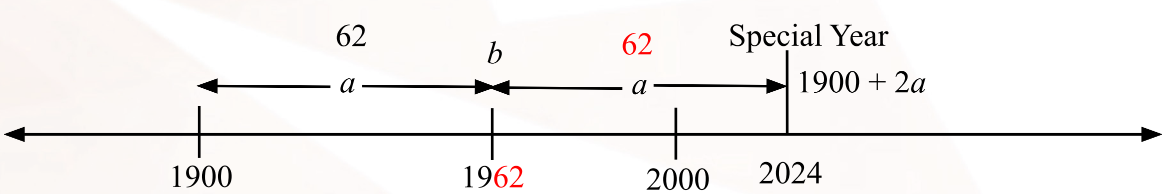


Figure 1. Special Year for Laurie

Also, if their Special Year is S, their age is $a = \frac{S-1900}{2}$.

Keywords: Age, pattern, relationships.

For a person born in 1900, their Special Year is their birth year, 1900. The Special Year of a person born between 2000 and 2099 is $S = b + (b - 2000)$. If a person born between 1900 and 1999 has Special Year S_1 and a second person born between 2000 and 2099 has Special Year S_2 then they share the same Special Year if $S_1 = S_2$, so $b_1 + (b_1 - 1900) = b_2 + (b_2 - 2000)$,

so $b_2 - b_1 = 50$, and thus $a_1 - a_2 = 50$.

In general, if the difference of the ages of two people born in consecutive centuries is 50 years, then they share the same Special Year. Thus, Laurie (as well as everyone born in 1962) and everyone born in 2012 share the same Special Year, 2024. Such are Special Years in the Common Era. (See Figure 2.)

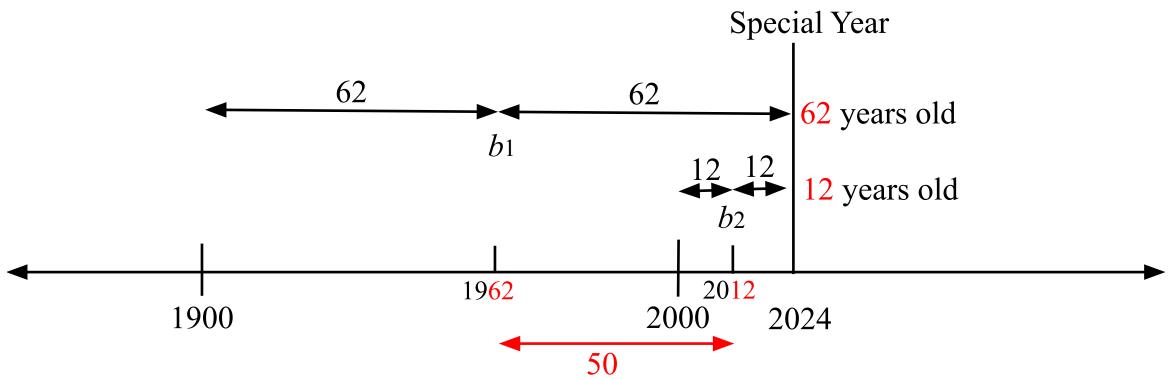


Figure 2. Shared Special Year

Pythagoras was born in the year 570 B.C.E. In his Special Year he was a years old, so the last two digits of his birth year $b - 500 = a$. Since $S = b - a$, by substitution, $S = 500$. Surprise! Not only Pythagoras, but everyone born between 599 and 500 had the same Special Year, 500 B.C.E. Of course, no one born before the Common Era would even know they had a Special Year unless they used a benchmark of year 1 C.E. as we do. (See Figure 3.)

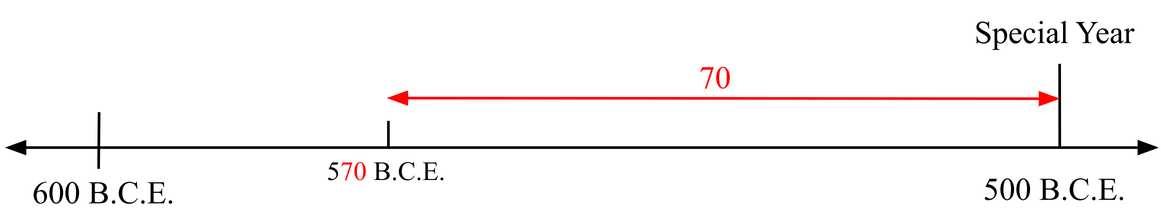
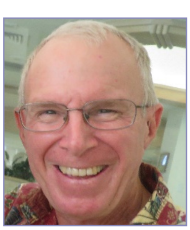


Figure 3. Special Year for Pythagoras



James Metz, is a retired mathematics instructor. He volunteers for Teachers Across Borders - Southern Africa for one month each year and enjoys doing mathematics. James may be contacted at metz@hawaii.edu

Brad Uy hails from the beautiful islands of Hawaii and is a creative and light-hearted educator. He has taught maths and English for 15 years, and has volunteered on 8 trips with Teachers Across Borders – Southern Africa. He currently teaches Algebra 1 at Kamehameha Schools Kapālama on O`ahu, an educational institution dedicated to serving native Hawaiian children. For recreation, Brad enjoys reading, tennis, cooking, and coffee. Brad may be contacted at Brad.uy@gmail.com

Review

Review of Algebra Tiles

MATH SPACE

Algebra tiles are a generalization of 2D base-10 blocks popularly known as flats-longs-units, or FLU, which were reviewed in the March 2024 issue of At Right Angles [5]. Simply speaking, the ten generalizes to "x" as follows, to form the three basic algebra tiles (with suggested dimensions: $x \to$ 2inch and $1 \to$ 2cm):

- The big square, i.e., flat or 100 becomes x^2 ($x \times x$ or 2inch × 2inch)
- The rectangle, i.e., long or 10 becomes x ($x \times 1$ or 2inch × 2cm)
- The small square, i.e., unit or 1 remains the same (1×1 or 2cm × 2cm)

But there is a crucial difference: algebra tiles come in two (contrasting) colours to represent positive and negative versions, i.e., x^2 and $-x^2$, x and $-x$, 1 and -1 as shown in Figure 1. Most of the virtual (and online) versions show the positive tiles in different colours based on size but all negative tiles in the same colour. Logically, if all negative tiles are in the same colour, then the same should happen for all positive tiles. Mathigon Polypad allows one to make the colours uniform (Figure 1). Another option is to make the tiles double sided so that one side represents a positive tile while the other represents a negative one. There are several advantages to double-sided tiles such as:

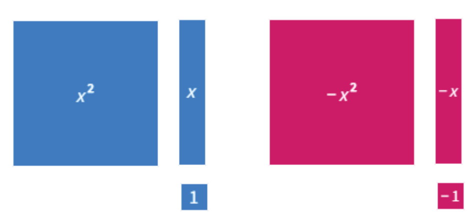


Figure 1

1. These are easier to make from any box with an exterior easily distinguishable from its interior.
2. Only one set of tiles need to be made, instead of two separate sets for positive and negative.

Keywords: *TLM, modelling, algebra tiles, representation, algebraic operations*

3. Pedagogically speaking, turning over or flipping a tile changes its sign from positive to negative or vice versa. So, flipping a tile is equivalent to sign change. This becomes very useful for subtraction.

Since algebra tiles include negative tiles, an understanding of integers is crucial (see [6]), especially:

A. Zero-pairs: 1 and -1, x and $-x$, x^2 and $-x^2$ which can be brought in and taken out as and when needed as they do not change the expression.

B. Subtracting a quantity is equivalent to adding its additive inverse, for example, subtracting 13, -7, $5x$ and $-2x^2$ are equivalent to adding -13, 7, $-5x$ and $2x^2$ respectively.

C. Positive × negative and negative × positive are negative.

D. Negative × negative is positive.

With these six tiles in three sizes, we can show any polynomial (i) in one variable, (ii) of degree 2, (iii) with integer coefficients. Usual algebra tiles do not allow fractional coefficients. Figure 2 represents several such polynomials with blue indicating positive tiles and pink indicating negative. Notice that $4 - x^2$ is actually $4 + (-x^2)$ and $5x - 1$ is $5x + (-1)$. This is why we need the negative tiles.

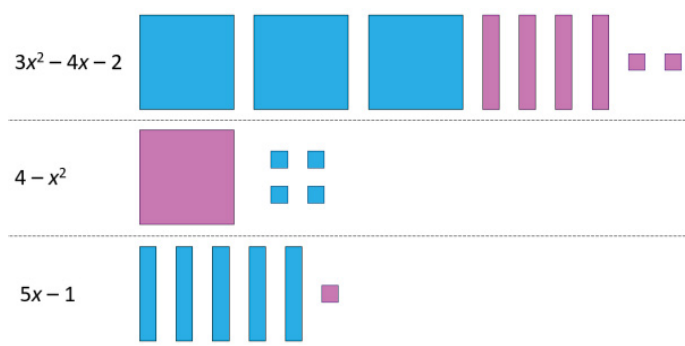


Figure 2

However, the tiles in Mathigon Polypad can be halved to get $x/4$, $x^2/2$, 1/8 etc. Only halving is allowed (horizontally or vertically). So, it is not possible to show $x/3$, $x^2/5$, 1/6 etc. (Figure 3).

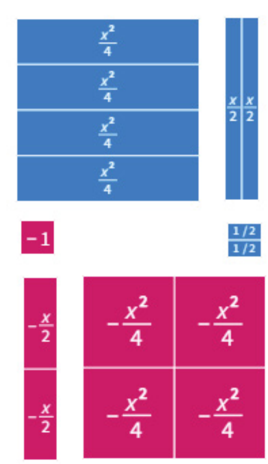


Figure 3

Another crucial difference between FLU and algebra tiles is the absence of exchange. In the case of FLU, it is a universal fact that 10 units make a long, and 10 longs make a flat. However, in the case of algebra tiles, since the value of x is not known or can have many possibilities, we do not know how many 1s are equivalent to an x. Therefore, there is no exchange among the tiles. This is similar to why the terms in a polynomial do not collapse into one term. So, the tiles reinforce the notion that polynomials like $3x^2 - 2x - 5$ cannot be simplified further.

As we represent polynomials using algebra tiles, a few things automatically become clear:

- Why x^2 is called x-squared and how it is linked to the geometric square.
- The difference between x^2, i.e., $x \times x$ as one square tile vs $2x$, i.e., $x + x$ as two rectangular tiles.
- The notion of like terms – one can count tiles of the same size, can even remove zero-pairs but cannot combine tiles of different sizes to be expressed as a single term.

Since the relation between x and 1 is unknown, there is no way to compare two polynomials. However, we can add and subtract any two polynomials as long as they can be represented by the tiles.

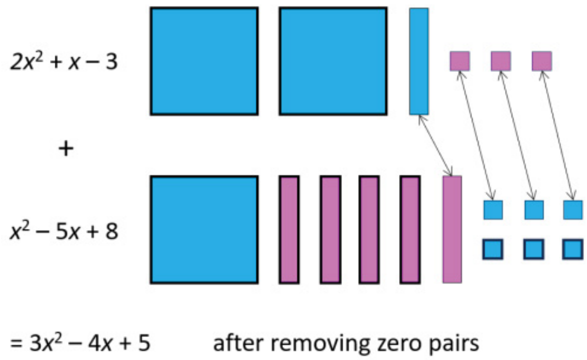


Figure 4

The rules of addition are similar to that for whole numbers with FLU:

1. Make each polynomial with the tiles.
2. Combine tiles of the same size and discard any zero-pair.
3. The remaining tiles represent the sum.

Similarly, the steps for subtraction, i.e., $p(x) - q(x)$ are:

1. Make the polynomials $p(x)$ and $q(x)$.
2. Flip the tiles of $q(x)$ to get $-q(x)$
3. Add $p(x)$ and $-q(x)$, i.e., combine tiles of same size and discard any zero-pair.
4. The remaining tiles represent $p(x) - q(x) = p(x) + [-q(x)]$

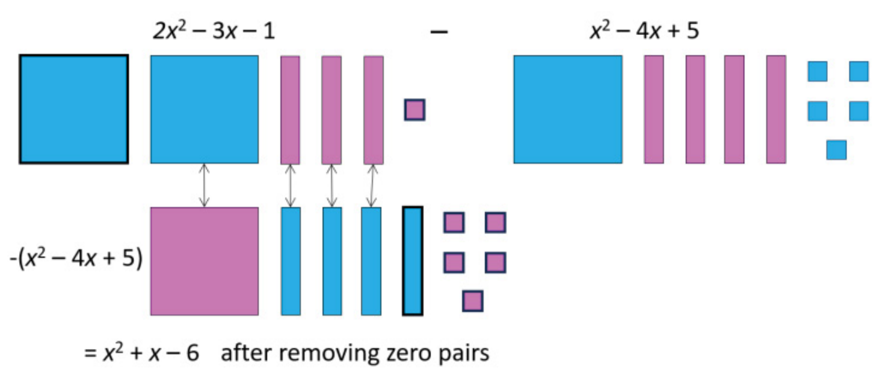


Figure 5

It is possible to execute a subtraction $p(x) - q(x)$ in a different way, that is closer to the whole number process:

1. Make the polynomial $p(x)$
2. Imagine the polynomial $q(x)$
3. Add zero pair(s) to $p(x)$ as needed so that there is enough to subtract $q(x)$ [i.e., remove x^2, $-4x$ and 5]
4. Subtract $q(x)$ from $p(x)$, i.e., remove tiles as mentioned above, what remains represents $p(x) - q(x)$

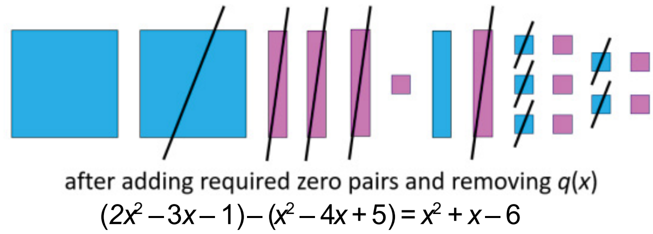


Figure 6

Figure 4 shows the sum of two polynomials and Figure 5-6 illustrate the difference of two polynomials.

However, this can be quite cumbersome since it may need zero pairs of more than one type (1 and -1, x and $-x$, x^2 and $-x^2$). Moreover, given B, i.e., subtracting a term is equivalent to adding its additive inverse, every subtraction can be considered as an addition. Therefore, flipping $q(x)$ is a simpler option for polynomial subtraction.

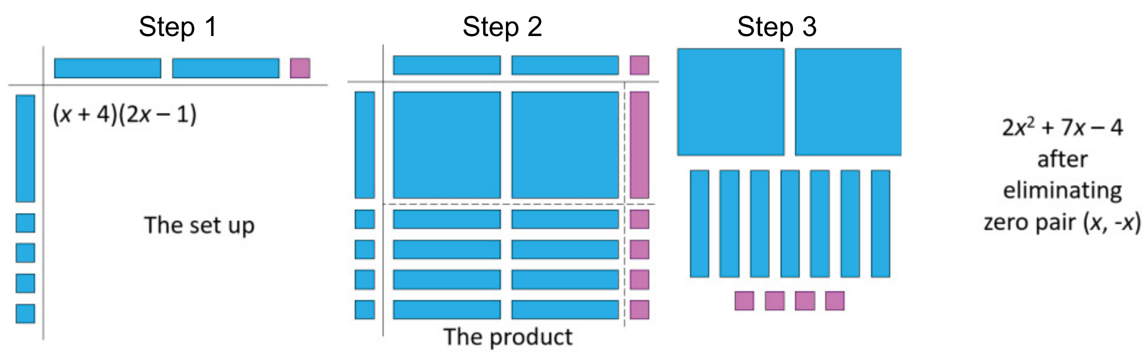


Figure 7

Multiplication-division are similarly restricted to polynomials of a maximum degree of 2. The product of two linear polynomials using algebra tiles is very similar to the product of two 2-digit numbers. Note the four regions of big and small squares and horizontal and vertical rectangles in Step 2: The product of Figure 7. The steps for multiplication are:

1. Arrange one factor polynomial along the left border and the other on the top border.
2. Fill the array, matching each dimension of each tile with the borders.
3. Remove any zero-pairs [Where can the zero-pairs occur in a product? Check Figure 8].

Figure 8 represents all possible combinations in terms of signs keeping the leading terms positive. Note the cases where zero-pairs occur. Consider the similarities and differences among these four cases. This understanding is crucial for middle term factorization. And yes, algebra tiles do a fantastic job of exploring that!

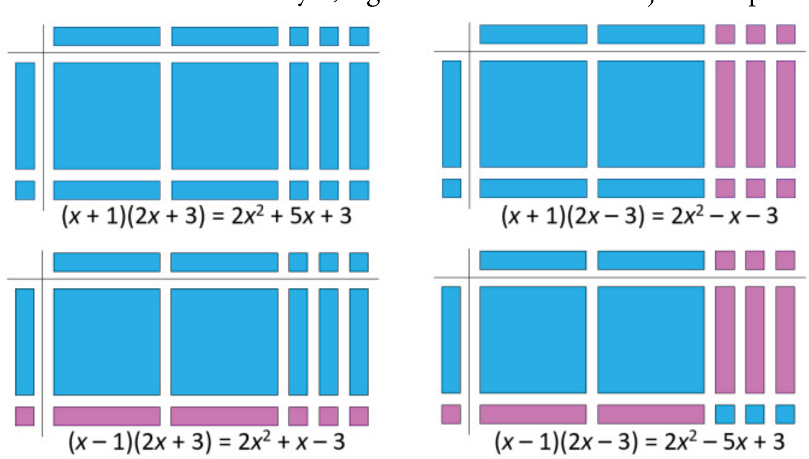


Figure 8

Division of a degree 2 polynomial by a degree 1 polynomial with algebra tiles (Figure 9) is also very similar to dividing a 3-digit number with a 2-digit number with FLU. See [5].

Step 1: Place the x^2 tiles to get the partial quotient $2x$ and complete the step by bringing in three zero pairs (x and $-x$), $3x - 3$ remains (Figure 10)

Step 2: Place the x tiles to get the remaining partial quotient 3 and complete the step with nine zero pairs (1 and -1), 9 is the remainder and $2x + 3$ is the quotient (Figure 11)

Note how the dividend is in two parts at the end – the array and the remainder, i.e.,

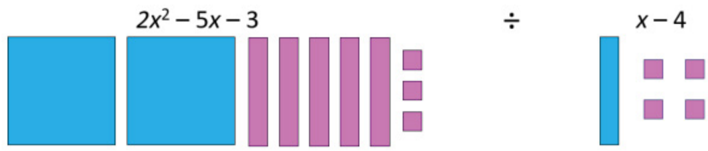


Figure 9

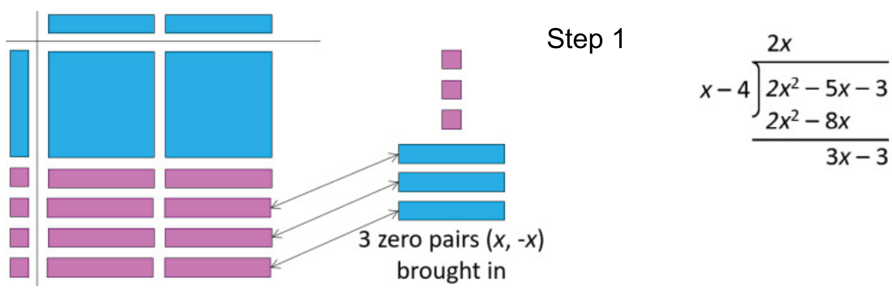


Figure 10

dividend = array + remainder = divisor × quotient + remainder just as FLU do (Figure 11)

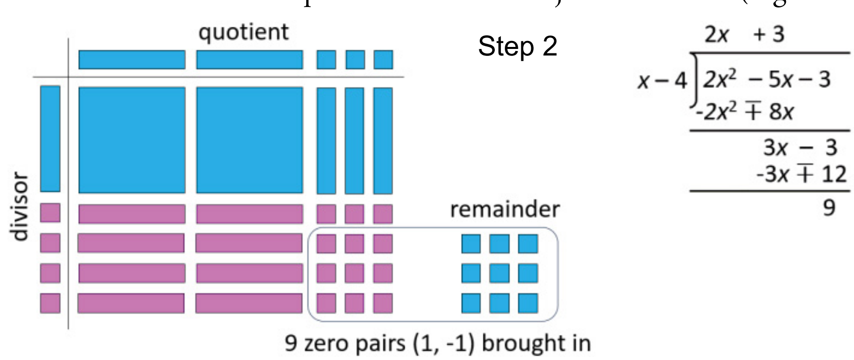


Figure 11

In addition, the algebra tiles can be used to represent all the quadratic identities:

- $(a + b)^2$ and related
- $(a - b)^2$ and related
- $(a + b)(a - b)$
- $(a + b)^2 + (a - b)^2$
- $(a + b)^2 - (a - b)^2$

So, as far as polynomials are concerned, algebra tiles help significantly. However, physical (or virtual) tiles have fixed dimensions. So, these are not great for equations. It may be possible to explore linear and quadratic equations in one variable with algebra tiles. We are yet to explore that in depth.

Some versions include one more variable y and tiles of three more sizes, viz., y^2, xy and y (Figure 12). This allows polynomials in two variables, but the remaining restrictions of degree and coefficients remain. We are yet to explore if these extended algebra tiles, especially, y along with x and 1, can provide some pedagogic advantage for say simultaneous linear equations in two variables.

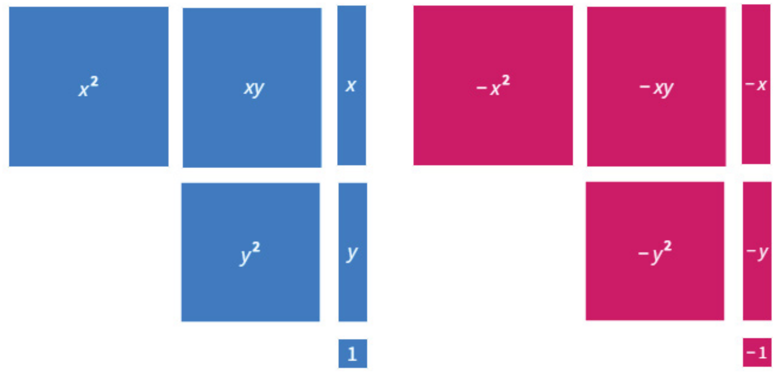


Figure 12

It is possible to address polynomials of degree 3 with cuboids. But it will be quite cumbersome and would definitely need two sets of cuboids – positive and negative – since the "flipping" advantage is lost with the 3rd dimension. Moreover, the cuboids may not provide additional pedagogic benefits to counter the effort.

Last but not the least, it is perfectly ok to introduce the tiles to students with no exposure to FLU. The tiles are a generalization of FLU and the transition from FLU to algebra tiles can help one see how whole numbers are simply 'polynomials in ten' with the digits as coefficients. But FLU is not a prerequisite for algebra tiles. However, coloured counters for integers are a prerequisite for the tiles (see [6]).

References

1. How to make algebra tiles: https://bit.ly/4buTsky
2. How to use algebra tiles: https://bit.ly/3zrFVNu
3. Explore algebra tiles virtually: https://bit.ly/3W7x3Wn
4. Algebraic identities with algebra tiles: https://bit.ly/3RSzyt0
5. FLU review: https://bit.ly/3XO1RMX
6. Integers: https://bit.ly/4bneWQw
7. Mathigon Polypad: https://bit.ly/3XLzU8o

MATH SPACE is a mathematics laboratory at Azim Premji University that caters to schools, teachers, parents, children, NGOs working in school education and teacher educators. It explores various teaching-learning materials for mathematics [mat(h)erials] their scope as well as the possibility of low-cost versions that can be made from waste. It tries to address both ends of the spectrum, those who fear or even hate mathematics as well as those who love engaging with it. It is a space where ideas generate and evolve thanks to interactions with many people. Math Space can be reached at mathspace@apu.edu.in

Flipping the Question Around!

If we consider different types of quadrilaterals – square, rectangle, rhombus, parallelogram, kite, trapezium, isosceles trapezium, dart (or concave kite) etc., we can easily state the kind of symmetry each type has. For example, rhombus and rectangle, each have a line of symmetry and rotational symmetry of order 2.

But what happens when we flip the question:

***A.** If a quadrilateral has line symmetry, what kind of quadrilateral is it?*

***B.** If a quadrilateral has rotational symmetry, what kind is it?*

Flipping a question provides scope for mathematical exploration, giving students opportunities to develop observation, documentation and analytical skills while honing their conceptual understanding. We invite responses from readers - send them to AtRightAngles.editor@apu.edu.in And we promise our take in the November issue!

A Call for Articles

At Right Angles is a quality resource dedicated to mathematics education in India's public education system. It is specifically designed for teachers and teacher educators at the foundational, preparatory, and middle school levels.

We invite articles from mathematics teachers, educators, practitioners, parents, and students. If you are looking for a platform to contribute articles that support and enhance the learning experience of mathematics particularly for students approximately in the age group 6-14 years, we welcome your submissions.

SUGGESTED TOPICS AND THEMES

Submitted articles should focus on curricular content applicable to Classes 1-8 and could:

- Explain and illustrate themes and topics outlined in the National Curriculum Framework for School Education 2023 (NCF-SE 2023).
- Specifically address challenges discussed in the NCF-SE 2023.
- Be substantiated accounts of the history of mathematics or the history of mathematical thinking.
- Include innovative worksheets or methods to engage students in drill and practice.
- Describe real-life applications of mathematics relevant to the child's context.
- Describe interdisciplinary activities or projects.
- Review puzzles or games with a practical connection to the syllabus.
- Offer guidance on selecting relevant content, including online resources.
- Develop pedagogical strategies for foundational numeracy as well as computational thinking.
- Assist teachers in implementing differentiated teaching practices.
- Review of Teaching Learning Material (TLM) or describe how to use local context, and local TLM in the math class.
- Provide material to help students bridge gaps in conceptual understanding.
- Address issues in assessment.
- Suggest ways to identify and address misconceptions in mathematics learning.
- Offer a list of problems along with discussions on their solutions and problem-solving strategies that are not commonly found in textbooks.

In addition to full-length articles, we also welcome shorter pieces that can include a variety of engaging content. These could be reviews of books, mathematics software, or YouTube clips that explore mathematical themes. Other contributions can be 'proofs without words', mathematical paradoxes, 'false proofs', or creative expressions such as poetry, cartoons, or photographs with a mathematical theme. We also welcome anecdotes about a mathematician or interesting examples of 'math in craft, movies, etc.'.

Articles may be sent to
AtRightAngles.editor@apu.edu.in

Please refer to specific editorial policies and guidelines on the following page.

Policy for Accepting Articles

At Right Angles is an in-depth magazine on matters of consequence to early mathematics and mathematics education. Hence articles must attempt to move beyond common myths, perceptions, and fallacies about mathematics.

The magazine has zero tolerance for plagiarism. By submitting an article for publishing, the author is assumed to declare it to be original and not under any legal restriction for publication (e.g. previous copyright ownership). Wherever appropriate, relevant references and sources will be indicated in the article.

At Right Angles brings out translations of the magazine in other Indian languages. Hence, Azim Premji University holds the right to translate and disseminate all articles published in the magazine.

If the submitted article has already been published elsewhere, the author is requested to seek permission from the previous publisher for re-publication in the magazine and mention the same in the form of an 'Author's Note' at the end of the article. It is also expected that the author forwards a copy of the permission letter, for our records. Similarly, if the author is sending his/her article to be re-published, (s)he is expected to ensure that due credit is then given to **At Right Angles**.

While **At Right Angles** welcomes a wide variety of articles, submissions that are found relevant but not suitable for publication in the magazine may be used in other avenues of publication within the University network, with the author's permission.

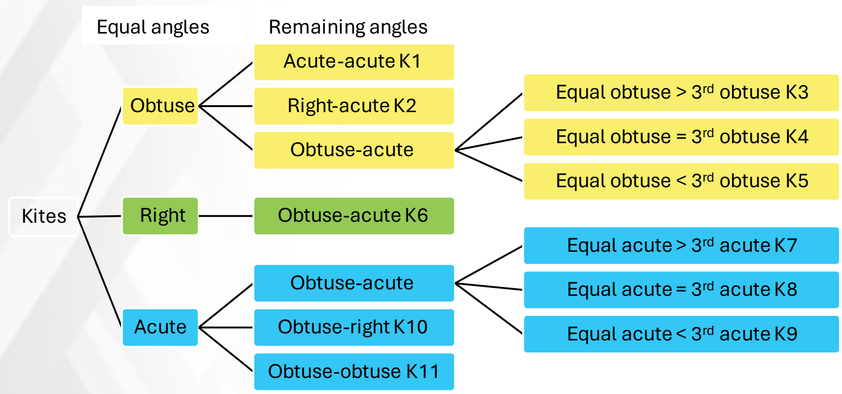
TearOut

Kite Families: An investigation of a family tree!

There are 11 types of kites (excluding rhombi) according to the poster.

This poster will help your students make friends with these 11 types. Do give the students time to study it and come up with properties for each of the kites K1, K2… K11. Some important points are given below. Students are sure to come up with these or other points during the discussion, if they don't then do share at your discretion.

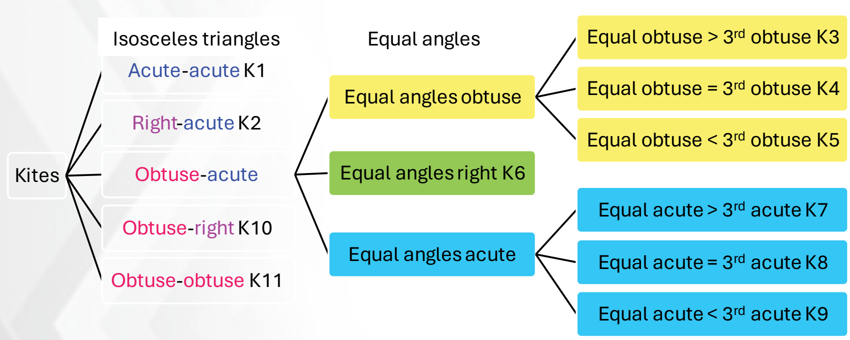
A. Along the line of symmetry



Two possibilities for rhombi:

i. Obtuse-acute, i.e., a non-square rhombus
ii. Right, i.e., square

B. As sum of two isosceles triangles



Rhombus: obtuse-obtuse or acute-acute depending upon the choice of diagonal
Square: right-right.

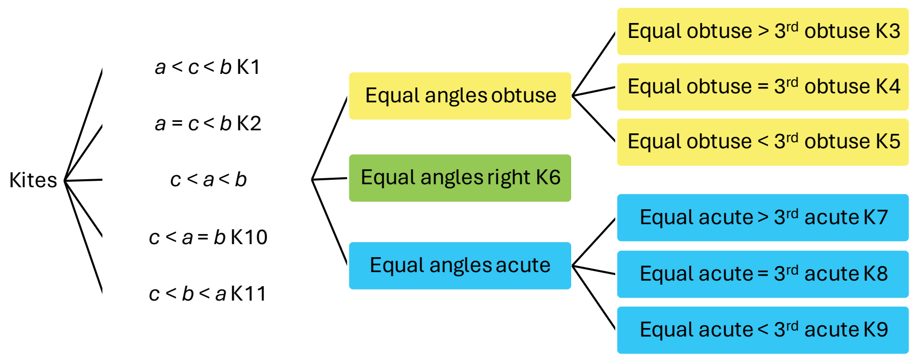
Keywords: *quadrilaterals, kites, properties, angles, classification, exploration*

C. Angle-wise

Classification identical to A.

K6 is the only cyclic kite with all four vertices on a circle.

D. Diagonal-wise



Halving diagonal longer for K1, K2… K6

Halving diagonal shorter for K8 (unless square), K9… K11

ABCD, PQRS, XYZW are K7 with equal acute angles > 3rd acute ones

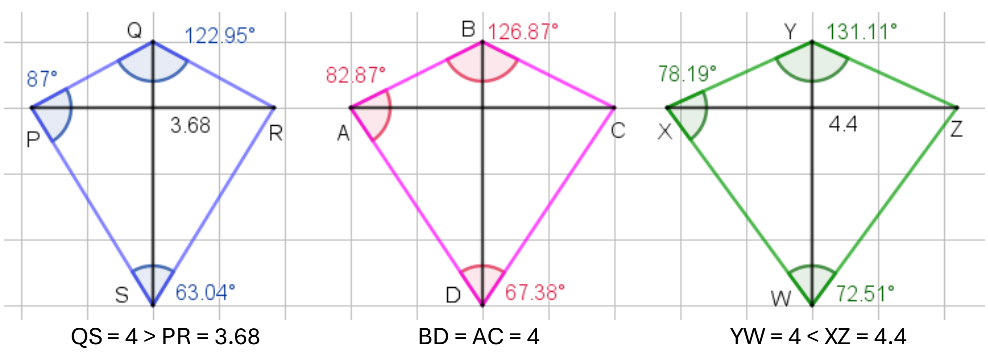


Figure 1

Proof:

Consider kite ABCD with AB = BC and AD = DC such that diagonals AC = BD intersect at O.

To show: ∠BCD < 90°

Construct circle with diameter BD.

Let the circle intersect OC at X. ∠BXD = 90° since it is an angle in a semicircle.

∠BCO + ∠XBC = ∠BXO (exterior angle equals sum of two opposite interior angles)

⇒ ∠BCO < ∠BXO

Similarly, ∠DCO < ∠DXO

∴ ∠BCD = ∠BCO + ∠DCO < ∠BXO + ∠DXO = ∠BXD = 90°

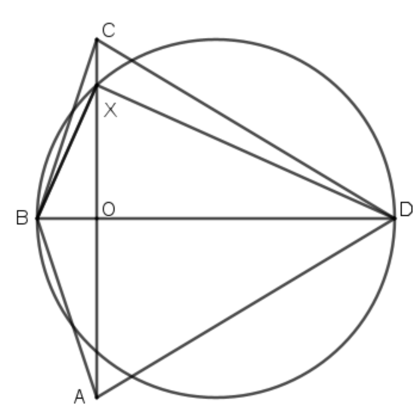


Figure 2

Specific Guidelines for Authors

Prospective authors are asked to observe the following guidelines.

1. **Engaging Introduction:** Write in a readable and inviting style, aiming to capture the reader's attention from the start. The first paragraph of the article should convey clearly what the article is about. For example, the opening paragraph could be a surprising conclusion, a challenge, a figure with an interesting question, or a relevant anecdote. Importantly, it should carry an invitation to continue reading.
2. **Catchy Title:** Title the article with an appropriate and catchy phrase that captures the spirit and substance of the article.
3. **Style:** Avoid a 'theorem-proof' format. Instead, integrate proofs into the article in an informal way.
4. **Balance:** Refrain from displaying long calculations. Strike a balance between providing too many details and making sudden jumps that depend on hidden calculations.
5. **Accessible language:** Avoid specialized jargon and notation that will be familiar only to specialists. If technical terms are needed, please define them.
6. **Use visuals:** Where possible, provide a diagram or a photograph that captures the essence of a mathematical idea. Never omit a diagram if it can help clarify a concept.
7. **Concise References**: Provide a compact list of references, with short recommendations.
8. **Exercises and Questions:** Make available a few exercises, and some questions to ponder either in the beginning or at the end of the article.
9. **Citation format:** Cite sources and references in their order of occurrence, at the end of the article. Avoid footnotes. If footnotes are needed, number and place them separately.
10. **Abbreviations and Acronyms:** Explain all abbreviations and acronyms the first time they occur in an article. Make a glossary of all such terms and place it at the end of the article.
11. **Labelling visual elements**: Label and number all diagrams, photos and figures included in the article. Attach them separately with the e-mail, with clear directions. (Please note: the minimum resolution for photos or scanned images should be 300 dpi).
12. **Precise references to visuals:** Refer to diagrams, photos, figures and tables by their numbers and avoid using references of these kinds: 'here', 'there', 'above', 'below', 'to the left', 'to the right'.
13. **Author Bio:** Include a high-resolution photograph (author photo) and a brief bio (not more than 50 words) that gives readers an idea of your experience and areas of expertise.
14. **British Spelling:** Adhere to British spellings – organise, not organize; colour not color, neighbour not neighbor, etc.
15. **Format for submission:** Submit articles in MS Word format or in LaTeX.

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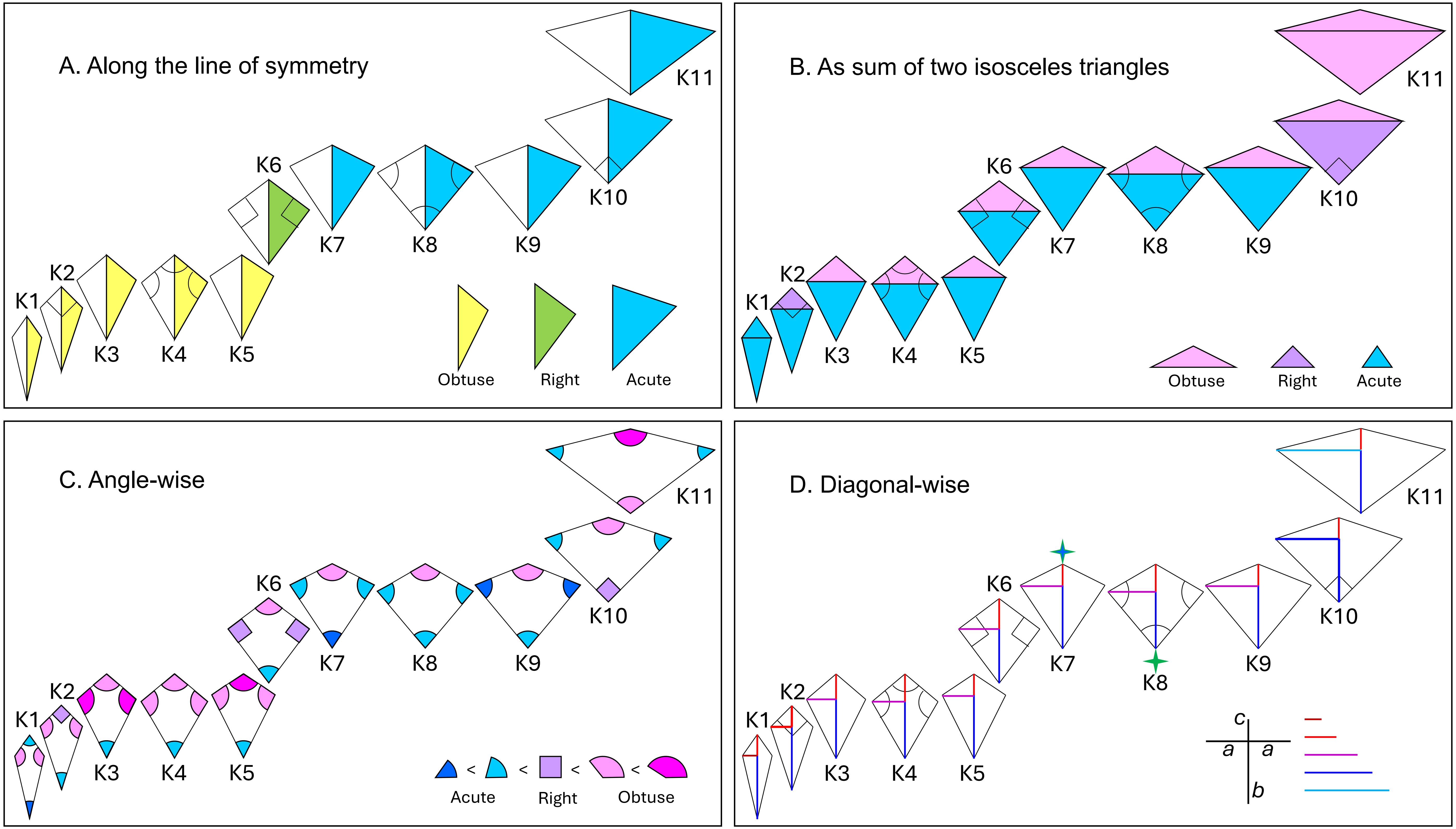
Kite Families



There are 11 types of kites (excluding rhombi) according to these pictures. K6 with 2 right angles is a special one, why? Can you characterise each of K1, K2… K11? Can you draw a kite for each of these 11 categories by specifying angles or sides-diagonals? Can you draw more than one for each category?

Is the halving diagonal (the line of symmetry) always the longer one? In "Diagonal-wise", the 2 categories marked with stars can have equal diagonals. The blue star category K6 includes kites of all 3 kinds (i) halving diagonal longer, (ii) equal diagonals and (iii) halving diagonal shorter. The green star category K8 includes kites with the halving diagonal being the longer one, and when the diagonals are equal, it becomes a square!

Here are some questions students can explore.

A. Along the line of symmetry

1. What is the line of symmetry in each kite?
2. Why is it the line of symmetry?
3. Consider the triangles formed by the line of symmetry. Based on these triangles, in how many distinct groups can you classify K1, K2… K11? What are these groups?
4. These groups categorize the pair of equal (and opposite) angles of the kite. Can there be subgroups within each group? What do these subgroups categorize?
5. Optional: Can you form a tree diagram?
6. If we consider rhombi instead of kites, how many possibilities are there considering the triangles formed by a line of symmetry?

B. As sum of two isosceles triangles

1. Take any kite. Consider the common side of the two isosceles triangles. How is this side related to the kite?
2. Consider all possible isosceles triangles – acute, right, obtuse. What are the possible combinations that generate:
   a. A kite
   b. A rhombus
   c. Any other special quadrilateral possible? Which one?
3. What would be the equal angles (acute/right/obtuse) of each such kite?
4. Classify K1, K2… K11 based on the classification in 2a. Can you find a kite outside K1, K2… K11?

C. Angle-wise

1. Consider the largest angle in any kite. What type of angle is it?
2. What type of angle is the smallest one?
3. A kite has a pair of equal and opposite angles. How many types of kites are there based on this pair?
4. Why does K2 have a light blue angle < the darker blue angle of K7?
5. Can you form a tree diagram classifying different types of kites based on the angles? Indicate where each of K1, K2… K11 are on this diagram.
6. Can you give examples for each of K1, K2… K11? E.g., K6: 120°-90°-90°-60°
7. Can you find a kite with an angle combination outside K1, K2… K11?
8. Draw a K6 kite. Draw a circle with the halving diagonal (or line of symmetry) as the diameter. What do you observe? Do you observe the same for K1, K2... K5 or K7, K8… K11?

D. Diagonal-wise

1. Describe each of K1, K2… K11 in terms of the parts of the diagonal a, b and c. E.g., K2: $a = c < b$
2. Draw a kite for each of K1, K2… K11 using your choice of a, b and c. Ensure $b > c$.
3. Which diagonal is longer for K1, K2… K6? Which one for K9, K10 and K11? Why?
4. Find the relation among a, b and c if the diagonals of a kite are equal.
5. Now draw 3 K7 kites as follows:
   a. Halving diagonal longer than the other one
   b. Both diagonals equal
   c. Halving diagonal shorter
6. Challenge: Prove that if the diagonals of a kite are equal then the equal (and opposite) angles must be acute.

MATH SPACE is a mathematics laboratory at Azim Premji University that caters to schools, teachers, parents, children, NGOs working in school education and teacher educators. It explores various teaching-learning materials for mathematics [mat(h)erials] their scope as well as the possibility of low-cost versions that can be made from waste. It tries to address both ends of the spectrum, those who fear or even hate mathematics as well as those who love engaging with it. It is a space where ideas generate and evolve thanks to interactions with many people. Math Space can be reached at mathspace@apu.edu.in

At Right Angles
A Resource for School Mathematics

AS HOT AS MATH? AS COOL AS MATH?

PADMAPRIYA SHIRALI

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Keywords: Temperature, experimentation, observation, documentation, analysis, lifestyle change

The content of the PullOut section has, so far, largely been topic based and focussed on mathematical activities and pedagogy. This particular PullOut is an attempt at integrating mathematics with science concepts in order to develop responsible ways of living.

The topic covered- Temperature -is appropriate for upper primary, class 6. The objectives are to understand the concept of temperature and the factors that affect it, discuss and experiment with ways of measurement of temperature and to notice its relationship to our daily living.

All over India we are experiencing hotter summers and unusual winters. Our day-to-day conversations include talking about the weather: the heat, rainfall, lack of water. Predictions about the weather and rain are eagerly being watched, discussed and anticipated.

The study of weather and temperature is topical for all of us during the current period. The topic provides scope for measurement and data collection and allows easy linkage with other subject areas: Geography (physical features that play a role); Physics (how heat is absorbed, radiated and conducted), data related to temperature rise, etc.

The study of temperature should incorporate observations, data collection, data representation and interpretation. It is worthwhile to study past data, to analyse records about factors that play a role and look for patterns.

Apart from conducting experiments to understand the concept of temperature and learning to estimate and measure temperature, it is important that the students begin to ask questions about everyday things that they take for granted or tend to overlook. Each question has the potential to lead to further questions that help in understanding the topic at a deeper level.

In this PullOut, I share varied experiments that can be conducted to develop an observation-based understanding of the concept, and raise questions to explore the topic further.

Expectation of prior knowledge

- Notion of temperature: Students should know about body temperature and how it is measured. They would have also experienced hot days and cold days and the temperature difference between summer and winter seasons.
- Ability to compare two/three objects of different temperatures and order them. Use proper vocabulary: boiling, very hot, hot, warm, cool, cold, very cold, freezing.
- Idea of average as a measure.
- Idea of a line graph and basic skill of reading a vertical/horizontal axis

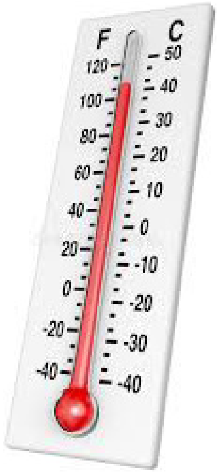
Concept: Temperature is the measure of the amount of heat present in a body, air, liquid or any substance. Explain to the students that temperature is the degree of hotness or coldness of an object. Children often confuse heat and temperature, regarding them as the same. The measurement of temperature is done in the Celsius system (named after a scientist). The short form used is °C.

Usage: Show students the usage of a thermometer. It is better to use an analogue thermometer rather than a digital thermometer for measurement activities. The bulb should only touch what is being measured. The thermometer should be held in place for at least one minute until the red material stops moving.

Caution: Clinical thermometers should not be used for measurement activities.

INVESTIGATION 1:

Objective: To learn to read a thermometer scale



Introduce the topic of how the summer affects all of us and let students express what they feel about it. It is bound to bring up numerous complaints and difficulties faced by them. On their own, students may wonder about aspects of weather and ask their own questions. If they do not, raise questions to help students come up with ideas for experiments.

What time of day is it the hottest? When does the temperature begin to rise? Does it rise evenly? How long does it stay that way? When does it begin to reduce? Is there any pattern from day to day?

Discuss these questions before they start doing any measurements. Let students record their ideas and assumptions about these questions.

Give students an analogue thermometer which records the air temperature. Place the thermometer vertically up. (Tape it to a sturdy box, if necessary).

Investigate what happens to the red liquid in the bulb of the glass tube over a period of time.

The students can form groups and each group can measure the temperature in turn.

Show students how to read the temperature on a thermometer correctly and the usage of the scale. Let the students decide at what intervals they will take the measurements and record them. Ideally, they should do it every 30 minutes.

Time	Estimated temperature	Actual temperature
9.00		
9.30		
10.00		
10.30		
11.00		
11.30		
12.00		

After each measurement they can write down their estimate for the next recording. By afternoon, they may predict a reduction in the temperature.

In the following math class, students can discuss the recorded data. Does the data match their earlier assumptions? Has the rise started earlier/later than expected? Is the maximum temperature recorded at noon time? Or later than noon? Do they see any pattern in the way the increase/decrease has happened?

What other data can we draw from this information? Will the difference of maximum and minimum temperature be of any use to us? To whom will it be useful? If someone plans to visit this place, how will they find this data useful? If we had to describe the temperature with an approximate figure, what figure would we use? Do students have an intuitive sense of average?

INVESTIGATION 2:

Objective: To understand the functioning of a thermometer

Mr. Mehta works in an aquaculture farm where they breed certain types of fish in artificial ponds. He needs to check the temperature of the water as algae blooms and also grows more quickly in warmer water. Growth of algae can cause a reduction in the amount of oxygen available in the water, causing problems for fish and wildlife.

What is the temperature of tap water (note: tap water can be quite hot in summer!) Fill a beaker with tap water and place the thermometer in it. What reading does it show? Perhaps around 25°C? Try with slightly lukewarm water.

Look at the thermometer scale closely. On the Celsius side, why do the markings go from 50 down to 0 and further to –40 degrees. What does that mean?

What happens to the red liquid? It rises and moves up the tube. Explain the functioning of a thermometer in simple terms. When the air around becomes warmer, the liquid expands, and it moves up the tube.

Try with cold water. What happens now?

Watch how the students conduct the investigation. Did they notice that the level of the liquid fell? Can they explain why? Were the explanations given with care and thought? Did the students discuss with seriousness, and did they listen to each other attentively?

Explain that when the air becomes cooler, the liquid cools and contracts.

Fact: Hot water from natural hot springs can have a temperature above 50°C and in some cases above the boiling point of water (100°C).

Let students reflect: What are the various situations where we use hot water at home? Do industries use hot water too?

INVESTIGATION 3:

Objective: To learn that water boils at 100°C and freezes at 0°C.

At what temperature does water boil? Freeze?

To check the temperature at boiling point, which needs to be done with a lot of care, use a laboratory thermometer. Students can use a kettle to boil water, pour out some into a beaker and check the temperature.

Students can be given ice cubes in a beaker. The temperature of the cubes will be close to 0°C.

INVESTIGATION 4:

Objective: To learn that answers can vary and are dependent on various factors.

How long does it take for a water bottle to freeze?

Let the students discuss the various factors that affect the situation before they estimate an answer.

Do they see that the starting temperature of the water will alter the time?

How about the volume of the water? How about the container which holds the water? What about the temperature of the freezer?

What are the various things that they can do to speed up the freezing?

Another question to investigate is: What is the temperature of water from a water cooler/fridge (cooled for a long time)? What is the temperature of water from a *matka*?

The clay pot (*matka*) has been a staple in Indian households. Some people think that it is better to drink water cooled in an earthen pot rather than in a refrigerator. Find out from a doctor if it is so. What are the reasons given by the doctor?

INVESTIGATION 5:

Objective: To observe the variation in temperature as the location varies

Is the temperature in a shady region (under a tree) less? By how much? What pattern does the temperature in a shady region follow? Is it always less than a sunny spot by the same number of degrees?

Does the temperature in the classroom follow a similar pattern? By how many degrees is it less than the outside temperature? Is it always less than a sunny spot by the same number of degrees?

Discuss: Which areas in the classroom are hot? Which areas are cooler? Why? Is there a difference between the temperature at the ground level and ceiling level? What could be the reason? What happens in summer when you put on a ceiling fan?

Let the students do the necessary measurements by selecting suitable places and recording the data. Students should display their recordings in the form of a line graph to contrast the different settings. What inferences do the students make? Are they able to give reasons for their inferences using the given data? How far did their predictions come true?

Location	Sunny spot temperature	Shady spot temperature	Inside classroom temperature
9 AM			
11 PM			
1 PM			
3 PM			

INVESTIGATION 6:

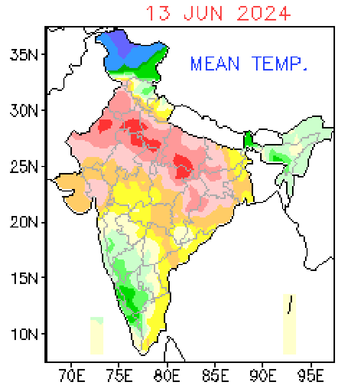
Objective: To explore the topic further by moving from local temperatures to temperatures across the country

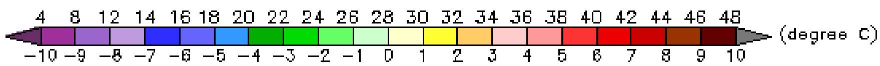
Do other places, say, capital cities of states (or other districts in the state) in India experience similar temperatures at this time? What cities are those? Which cities have higher temperatures? Which cities have lower temperatures? What could be the reason for the difference?

Use a visual or data table containing information about the temperatures of different cities.

Average temperatures in various Indian cities (°C) (https://en.wikipedia.org/wiki/Climate_of_India)

City	Summer (Mar – May) Min	Avg	Max
Delhi	16	28	36
Port Blair	25	27	29
Thiruvananthapuram	24	27	30
Bangalore	21	27	34
Nagpur	24	32	40
Bhopal	23	30	36
Guwahati	19	25	31
Kolkata	24	30	35
Lucknow	23	30	35
Siliguri	19	25	31
Jaisalmer	24	33	40
Dehradun	14	23	32
Amritsar	13	25	34
Shimla	10	14	18
Srinagar	7	14	19
Leh	−1	6	12





https://mausam.imd.gov.in/ClimateInformation/imdweb/DAY_WEEK/t-0.gif

Students should mark these places and record the information as an exercise on a map and study it.

Let students notice the various aspects of this data. Encourage them to formulate questions. Raise some guiding questions. Are there similarities in the coastal areas? What physical features of India can be seen in many of the regions marked green? Is there a presence of hills? Forests in these regions? In which places is the difference between the minimum and the maximum the highest? Which part of India are they located in? What physical features can be the reason for Jaisalmer's temperatures?

Facts: The highest temperature ever recorded in India occurred in May 2016 in Jodhpur District, Rajasthan at 51.0°C. The lowest recorded temperature in India was –45.0°C in Ladakh in January 1995.

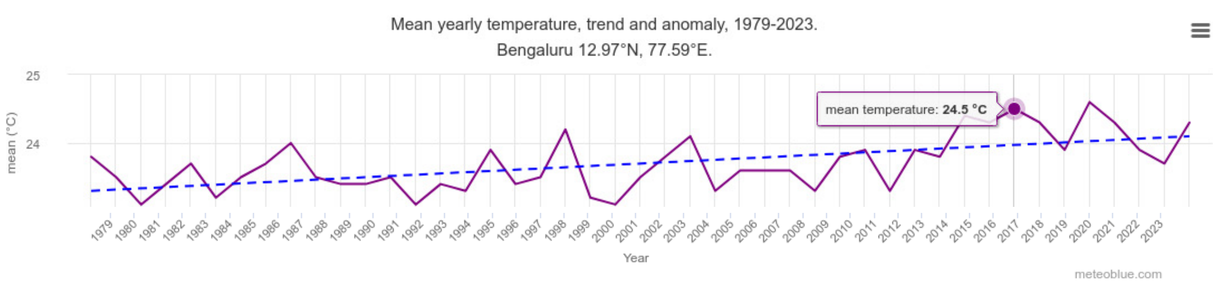
Mark these places on the map. Is there anything common to both these places?

INVESTIGATION 7:

Objective: To notice and analyse the changes that happen over time

Ask the students about stories that they have heard from their elders about their experience of summers. Most students would have heard their parents, grandparents talk about their younger days when they could manage without the need for air conditioners, coolers, refrigerators, etc. They would have heard stories of being able to sleep on a terrace or on verandahs under the open skies.

Pose the question: How have the temperature measures changed over the years in our region?



https://www.meteoblue.com/en/climate-change/bengaluru_india_1277333

Use data of a nearby city and let students make their observations about their findings. While this particular graph also shows the trend line, the average (mean) temperature graph is adequate to notice how the yearly temperatures are rising.

In which years has the purple line crossed 24°C? Is the rise happening frequently in the past few years? What could be the cause of this rise? Does it have to do with the rise in population? How would that affect the temperature?

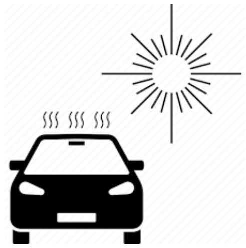
Does it have to do with industrial pollution? Does it have to do with excessive constructions?

Does it have to do with the usage of cement, glass in our constructions?

Note: Research and find out how overpopulation, industrial pollution, concrete constructions affect the temperature.

INVESTIGATION 8:

Objective: To understand the effect of heat on metal.



Most of us have experienced the discomfort of getting into a car, bus or a railway engine that has been parked in the sun.

Pose the question: How do various materials respond to temperature rise? How hot does it get in a car?

Students can use a tin box, check the temperature inside the box before placing it under the sun. Let them measure the temperature every five minutes and record the results.

Discuss how quickly the temperature inside the box has risen. Talk about the need to stay hydrated while traveling in or waiting for buses or trains, particularly during the summer.

Facts: After just 20 minutes on an average 26-degree day, the inside of a parked car can hit 42°C. After 40 minutes, the temperature can touch 47°C.

INVESTIGATION 9:

Objective: To understand the significance of water temperature for aquatic life.

Discuss how water temperature affects aquatic life. Certain types of fishes and aquatic life thrive well in the sea and some thrive well in lakes. What is the difference? Is the ocean water warmer than the river water?

Find out: About the temperature of water that is needed by fishes and whales, etc.

Fact: The Indian Ocean basin is the warmest ocean basin on the planet and the temperature can reach up to 28°C.

INVESTIGATION 10:

Objective: To observe the temperature at which beverages are consumed.

We all enjoy beverages and hot chocolate or hot coffee might be an interesting drink to study. At what temperature can we have these drinks without burning our mouths? At what temperature do they taste good? How long do I need to wait for boiling milk to cool down before I can have it with cocoa?

Fact: Incidentally, discuss whether the boiling point of milk is the same as the boiling point of water. Boiling point of milk is very close (100.5°C)

Let students measure the temperature of boiling water (milk is difficult in a classroom, but a kettle of water can be easily heated in a classroom) to test whether it is close to hundred degrees. Let them check the temperature every 5 minutes to find out how the temperature decreases. Ask the students to record the results on a line graph showing time in seconds (horizontal axis) and the temperature readings in °C (vertical axis). What do they notice about the graph?

Find out: At what temperature are biscuits/cakes baked?

Fun activity: Let students name a few places that they consider as the warmest spot in the school. In each of these places they can place an ice cube and measure the time that it takes to melt.

INVESTIGATION 11:

Objective: To understand how quantity affects heat loss.

Does a large amount of water lose heat faster than a small amount of water?

Pour hot water into a large container and a small container. Check the temperature of both the containers every 5 minutes. Plot a graph. How do the two graphs differ?

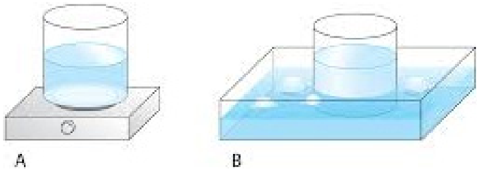
INVESTIGATION 12:

Objective: To understand heat transfer.

Raise the question of how hot things can be cooled quickly. Most students would have noticed their parents placing boiled potatoes in cool water or a hot glass of milk in a vessel of cold water to bring down the temperature.

Different questions can be raised for experimentation purposes. How long does it take for a cup of boiling hot water to decrease by 20°C? By how much does the temperature reduce if you keep a hot cup in ice cold water for 5 minutes?

Place warm water in a cup (A) and check the temperature. Place the cup with the thermometer in a basin containing cold water and ice cubes (B).



Let the students note the change in the temperature of the warm water.

Ask the students what they notice about the temperature of the water? Is it getting warmer or colder, or staying the same?

What is happening to the cold water? Is it getting warmer?

Point out how heat is being transferred from a warmer object to a colder object.

If the heat of the water has to be retained in a tumbler what are the various things that they can do?

INVESTIGATION 13:

Objective: To understand the effect of colour on heat absorption.

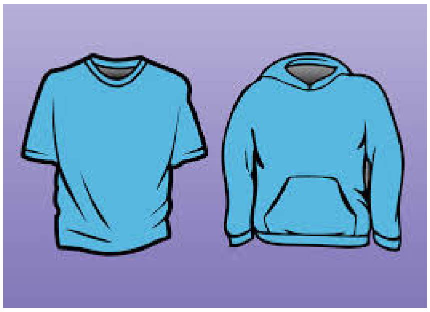
What effect do various colours have on heat absorption?

Students can fill 4 jars of the same size with equal amounts of water, cover them with different coloured papers (black, pink, blue and white) and place the jars in a sunny spot. They can make a prediction about which tumbler will hold the warmest water and which will hold the coldest. Did their prediction come true?

Discuss a related observation about why people wear light coloured clothing in the summer (corroborated by the earlier experiment) and dark clothing in winter. Explain that darker colours absorb more sunlight than lighter colours. Hence, darker colours get warmer more quickly in the sunlight than lighter colours.

INVESTIGATION 14:

Objective: To understand the function of clothing in retaining temperature.



Discuss what is best to wear on a cold day. A sweater or a T-shirt? Students will readily agree that they would wear a sweater.

Raise the question 'Is the sweater or T-shirt warm in temperature? If the sweater is not warm, how does it help to keep you warm?'

Give the students cues about body heat, the effect of cold air, how the sweater provides a protective layer.

Human beings use sweaters or jackets to retain their body warmth. What do animals do? Have they evolved natural coats and jackets? Students would have noticed how some animals are very hairy, some have fluffed up feathers, some are covered with wool. Discuss how thick layers of fat protect some animals from cold weather.

INVESTIGATION 15:

Objective: To notice the heating and cooling implements used at home.

What are the various ways of preserving heat in solids and liquids? Let students bring up examples of appliances that they use at home. (Geysers, Hot Water bottles, Thermos flasks, etc.)

Encourage the students to formulate questions about these appliances. For example, What is the maximum temperature of a geyser? What gradations of temperature are given on an iron? etc.

Students can find out how people preserved fruits and vegetables in the past. What techniques were used in constructions to keep a house warm/cool?

Fact: Liquid mercury freezes at -39°C temperature and boils at 357°C temperature.

INVESTIGATION 16:

Objective: To become aware of places with extreme temperatures.

Discuss how people cope with summer heat in different ways. Some people like to travel and where do they go?

Most people like to visit cooler places during summer. People in South India may like to go to Ooty or Coonoor. If they are in the northern part of India, they may go to Uttaranchal or Himachal. If they are closer to the North-East, they may visit Sikkim or Darjeeling. The temperature in Ooty may be below 25°C, in Himachal, it may range between 20°C and 25°C, Sikkim may be around 22°C. If people go high up to the foothills of the Himalayas or to Gangotri (where the Ganga river starts from), the temperature may be -1°C. They will need warm clothing! Definitely, they will experience much relief from the scorching weather of 35°C to 45°C in most places of India at that time.

Hardly a place to visit! But who would want to visit these places?

- Eastern Antarctic Plateau, Antarctica (-94°C)
- Alaska, United States of America (-73°C)
- Greenland (-69.6°C)
- Siberia, Russia (-67.7°C) (Schools shut down only if it's colder than -55°C!)
- Yukon Territory, Canada (-62.8°C)
- Alaska, United States of America (-62.1°C)

How would a line graph for this data look, if it is arranged in descending order of temperatures? Will it be going down from left to right or rising from left to right?

If the temperatures rise by 15°C in each of these places, how would the line graph change in appearance?

If the temperatures fall by 2°C in each of these places, how would the line graph change?

Here is a list of the top ten hottest places in India. What is the difference between their temperatures? Do you live in any of them? What measures do people take to protect themselves against heat?

Find out: Names and temperatures of the five hottest places in the world.

Top Ten Hottest Places in India

Sno.	Place	Maximum Temperature
1.	Delhi	44°C
2.	Churu	50°C
3.	Sri Ganganagar	50°C
4.	Bilaspur	49°C
5.	Nagpur	48°C
6.	Banda	48°C
7.	Vijaywada	45°C
8.	Jhansi	47°C
9.	Titlagarh	45.5°C
10.	Phalodi	51°C

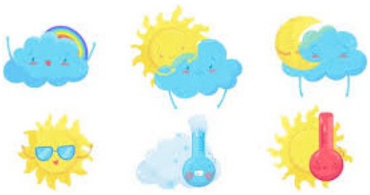
Project: Be a meteorologist!

Objective: To help students learn to build tables and records over a period of time in a systematic manner for finding patterns and doing analysis.

Discuss what a meteorologist does, how they gather information and look for patterns. Students should be made into groups to record and describe daily temperatures at specific times. They can also record other weather markers. Based on the patterns they observe, they can make predictions for the succeeding day or week. They can contrast their prediction with the standard weather reports. How close were their predictions to the actual?

Ask students to start recording/ describing everyday weather in a chart. Use the measured temperature and words such as sunny, windy, cloudy.

Compare everyday weather forecasts given in the newspaper/weather website for a week with the actuals. (make a graph of both to contrast the two figures.)

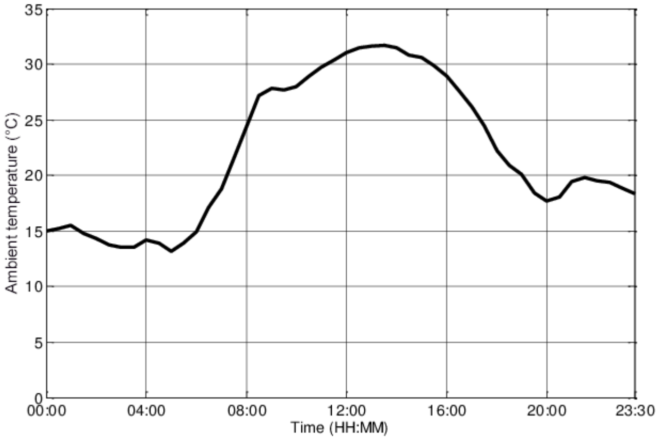


Discuss: What are the situations where we need exact temperature measurements?

INVESTIGATION 17:

Objective: To interpret temperature graphs

The graph shows what happened to the temperature during day and night. At what time did it begin to rise? At what time did it begin to fall?

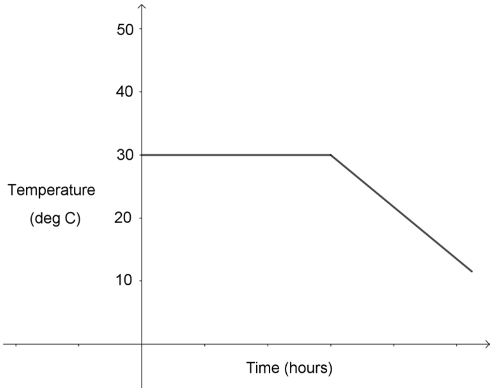


Which of the following graphs shows temperature being constant?

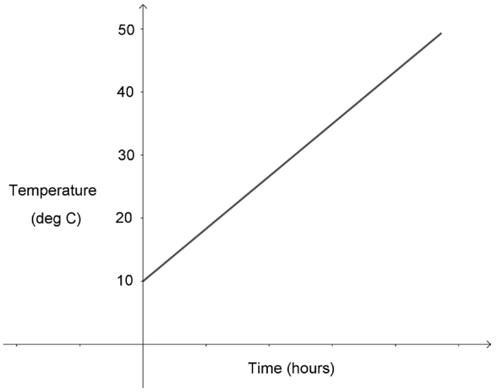
Which of the following graphs shows temperature being constant and then getting colder?

Which of the following graphs shows the temperature getting warmer?

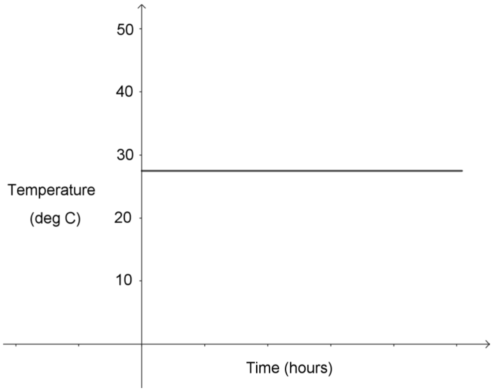
Which of the following graphs shows temperature being constant and then getting warmer?



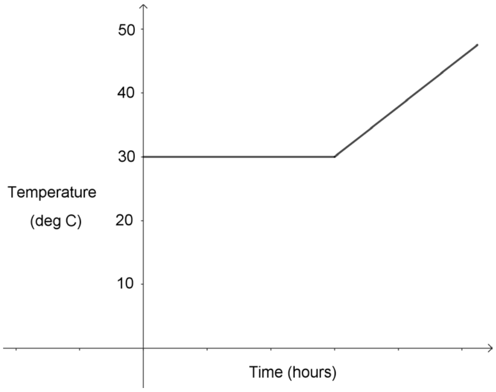
Graph A



Graph B

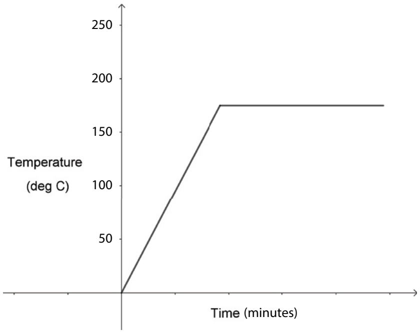


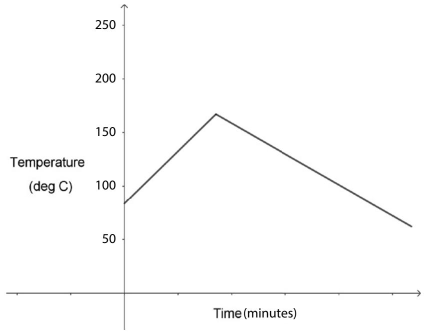
Graph C



Graph D

Ruhi preheats her oven to make a cake. Once the oven is warm, she puts the tray in the oven for baking. Which of these graphs shows her oven temperature during this process?





Discussion: Objective: To understand personal responsibility. ‘What can we do? How can I help?’

Many factors contribute to the rise in overall temperatures in the world. Teachers should discuss the topic of global warming at age appropriate level with the students.

It may not be possible to do much about them at a young age. But one can begin to do small but important things so that we do not contribute to the problem.

Turn off lights, TVs, computers when you are no longer using them!

Unplug any electronic device that you can turn on with a remote (TV, DVD player, laptop etc.). These devices use power even when they are “off.”

Reduce the usage of AC. When it is hot, use fans, which use less energy. When it is cold, wear warm clothing to conserve energy.

Walk, or ride your bike instead of taking a car everywhere.

You can help by growing your own vegetables and fruits.

You can help by planting a tree.

Use reusable grocery bags.

Recycle everything you can.

Use less paper whenever possible.

Drink filtered water instead of bottled water. Carry your drinking water in a reusable bottle. Plastic water bottles are an environmental disaster!

Buy the product that uses less packaging material. Even if you recycle packaging materials, it takes energy to create them in the first place and energy to remake them into something else.

A study of topics like Temperature should encourage experimentation, documentation of results, and analysis of data and should lead to understanding that serves as a propellor for changes in one’s lifestyle.

TEMPERATURE QUILT

A temperature quilt is usually a year-long project that can be undertaken by students of a class, especially if they are doing a study of temperature along the lines suggested in this PullOut.

Each day, you make a block depicting the high and low temperatures of that day. The choice of colours is entirely up to you, so is the block pattern. The blocks can be sewn by hand or with a sewing machine. The range of temperature is decided based on the place in which the data is being documented. The high and low should be recorded at the same time each day.

Figure 1

Quilter Chitra Lakshminarayan who is based in Chennai, chose the colours shown in the chart in Figure 1.

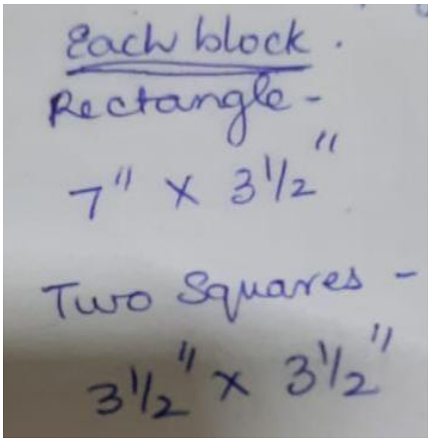


Figure 2. Each day's block was a 3.5 inch by 7 inch rectangle, composed of two 3.5 inch by 3.5 inch squares.

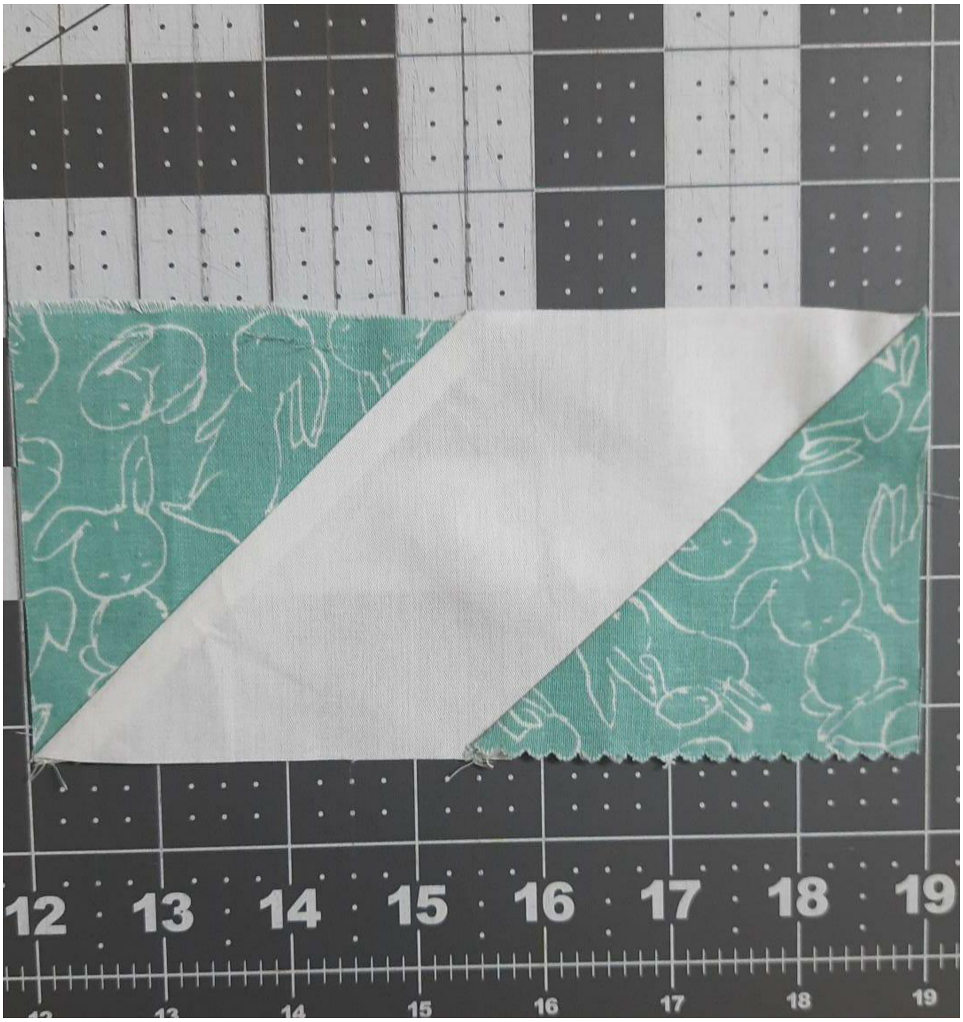


Figure 2

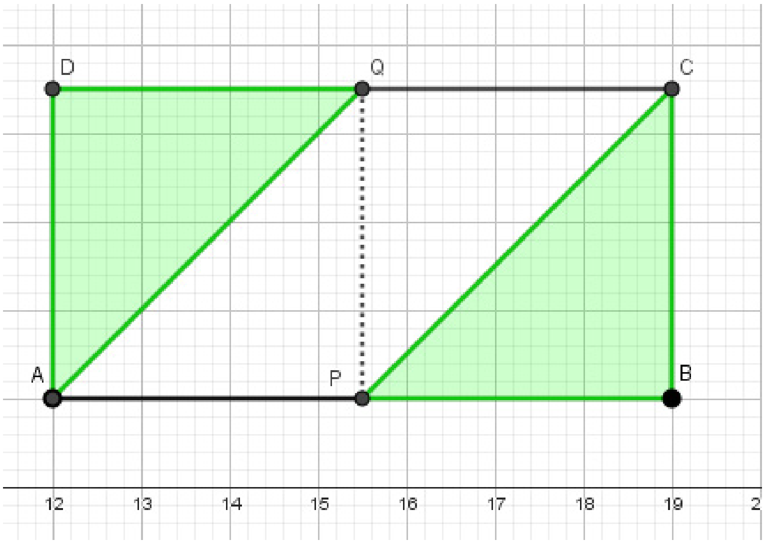


Figure 3

The parallelogram AQCP in the centre shows the low of the day and the triangles ADQ & PCB at the corners show the high of the day. How many congruent triangles do you see? The quilter begins by cutting 3.5 inch by 3.5 inch squares in each of the two colours (green and white) and then making half-square triangles which are then sewn together to make squares with contrasting triangles. The squares are then joined to make a 3.5 inch by 7 inch rectangle!

Finally, the rectangles for each month are sewn together to make a strip and then the strips are combined at the end of the year to make the beautiful quilt shown in Figure 4.

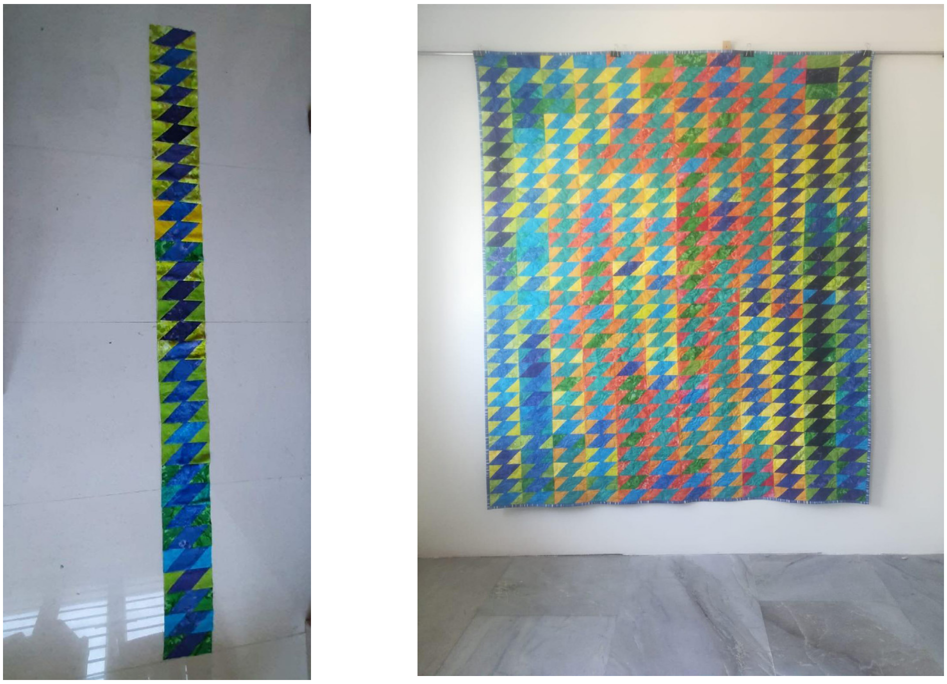


Figure 4

Figure 5a

Figure 5b

Figure 5 shows two temperature quilts made by Geetha Srinivasan, she documented temperatures in Chennai (Figure 5a) and in Sheffield, UK (Figure 5b). Notice the difference in the choice of size of blocks, geometry patterns in the blocks as well as colours.

Temperature quilts are an innovative way to document data, giving scope for creativity and innovation. Quilters are historians too!

Note from Editor: All the quilts shown were quilted at The Square Inch, Chennai.

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Student Corner

Aryabhata and the Construction of the First Trigonometric Table

VIJAY A. SINGH & ANEESH KUMAR

Introduction

We routinely use our calculator to obtain the sine of a given angle. (In earlier times we used published tables!) Little do we realize that a debt of gratitude is owed to the fifth century Indian savant Aryabhata who was the first to tell us about the sine function and create the first table of sines — in his seminal work, the *Aryabhatiya* (499 CE). In this article, we describe the trigonometric identities used by Aryabhata to obtain the table of sines.

The Indian mathematical tradition is largely word-based. Results are mentioned but derivations are omitted. The *Aryabhatiya* with more than 100 cryptic, super-compressed verses of dense mathematics is a prime example. In order to make our presentation pedagogical we take a unit circle and measure angles in degrees or radians instead of the (now) archaic notation in the *Aryabhatiya* and its commentators. We show that Aryabhata's sine table entails taking the difference of the sine of two closely spaced angles and then taking the second sine difference. We also show that the trigonometric identities are the same as those in the finite difference calculus one uses nowadays to numerically obtain the first and second derivatives of the sine function. We follow this with a brief discussion. An understanding of these identities and preparation of the sine table will enable the reader to get an appreciation of the path-breaking work of Aryabhata. At the end, we suggest a few problems and invite the reader to try their hand at solving them.

Keywords: Aryabhata, Sine, Sine Table, Finite Difference Calculus

The *Aryabhatiya* consists of 121 cryptic verses, dense and laden with meaning [1, 2]. The work is divided into 4 parts or *padas*: the *Gitikapada* (13 verses), *Ganitapada* (33 verses; the mathematics part), the *Kalkriyapada* (25 verses) and *Golapada* (50 verses; the astronomy part, which is better known than the others). There are two verses in *Ganitapada* describing the solution of the linear Diophantine equation. This has received due recognition. Our focus will be on the trigonometry part in *Ganitapada* which in our view has suffered neglect.

Virtually every major Indian mathematician has commented on the *Aryabhatiya*. Often it is in terms of a formal *Bhashya* (Commentary). Table I lists some of them. Notable among them is the voluminous work *Maha Bhashya* of the 15th century mathematician Nilakantha Somaiyaji (1444–1544 CE). He was part of the Kerala school which, beginning with Madhava (1350–1420 CE), founded the calculus of trigonometric functions.

Bhaskara I	629 CE	Sanskrit	Valabhi, Gujarat
Suryadeva Yajvan	1191 CE	Sanskrit	Gangaikonda-Colapuram
Parameshvara	1400 CE	Sanskrit	Allathiyur, Kerala
Nilakantha Somayaji	1500 CE	Sanskrit	Trikandiyur, Kerala
Kondadarma	unknown	Telugu	Andhra
Abul Hasan Ahwazi	800 CE	Arabic	Ahwaz, Iran

Table 1. A host of eminent mathematicians have commented on the *Aryabhatiya* written 499 CE. Some, like Brahmagupta (600 CE) or Bhaskara II (1100 CE), have not written a specific commentary but have dwelt extensively on it. The above is an abbreviated list of specific commentaries and the dates are approximate. Our main source is the work of K. S. Shukla and K. V. Sarma [2] which cites around 20 commentaries.

The *Ardha-Jya* or Sine Function

Aryabhata lays down — for the first time in the history of mathematics — a definition of the sine function. He poetically describes the sine function as the half bow-string or the *Ardha-Jya*, and relates the cosine functions to the arrow or *saar* (see Figure 1). This is not the only example of poetry making an appearance in his mathematics. To describe the fact — heretical and revolutionary for those times and for long afterwards — that the Earth is rotating and the Sun is stationary, Aryabhata evokes the tranquil metaphor of a boat floating down the river and the stationary river bank which seems to move backwards. Also being a poet, Aryabhata composed the *Aryabhatiya* in verse form with over 100 verses, respecting the norms of grammar and metre.

The sine function is the half-chord AP of the unit circle in Figure 1:

$$\sin\theta = \frac{AP}{OA} = AP \qquad (OA = 1).$$

The circle may be large or small; correspondingly, AP and OA may be large or small, but the LHS is a function of θ and is not dependent on the scale of the figure. All metrical properties related to the circle can be derived using trigonometric functions and the Pythagorean theorem (also known as the 'Baudhayana' or 'Diagonal' theorem [4]). For example, the geometric properties of a triangle can be related to the arcs of the circumscribing circle using the sine and cosine functions. Or the diagonals of the inscribed quadrilateral can be related to its sides. (A recent proof of the Pythagorean theorem using the the law of sines suggests that all metrical properties of a circle can by obtained using trigonometry alone [5].) By emphasizing the role of the half-chord, Aryabhata endowed circle geometry with metrical properties. This alone should qualify him as the founder of trigonometry. But he did more.

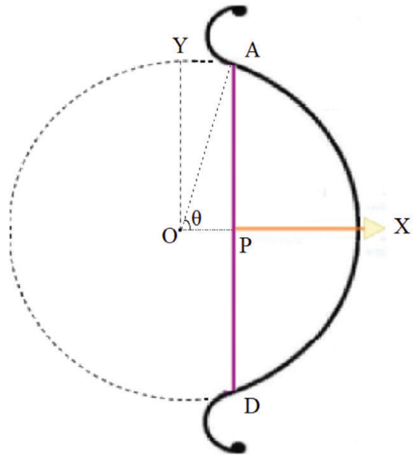

Figure 1. The bow superposed on the unit circle. Half the bow string or half chord AP is $\sin\theta$ as defined by Aryabhata. OP is $\cos\theta$, while $PX = 1 - \cos\theta$ is called the *saar*. See text for comments.

Note that the length of the half-chord AP is close to the length of the arc AX when the angle is small (i.e., $\sin\theta \approx \theta$ if θ is small and in radians). This was known to Aryabhata. Similarly, $\sin 90^\circ = 1$, since then the half-chord is the same as the radius. In the 9th verse of *Ganitapada* he uses the property of an equilateral triangle and obtains $\sin 30^\circ = 1/2$.

We pause to note that Aryabhata also states the value of π as 62832/20000 in the 10th verse. This value is 3.1416 and he is careful to state that this is 'proximate' (*Asanna*) suggesting that we can obtain better values for π with more effort. Here, the word *Asanna* or 'proximate' is to be distinguished from *Sthula* which is approximate or roughly equal.

The Difference Formula for Sine and Cosine

The 12th verse of *Ganitapada* plays a central role in the tabulation of the sine function. It is cryptic and to unravel its meaning we first need to obtain the difference formula for the sine. The presentation below relies on a number of sources: (i) The commentary of Nilakantha Somaiyaji [3]; (ii) the treatment of Shukla and Sarma [2]; (iii) and for the sake of ease of understanding we follow Divakaran [4] and work with a unit circle rather than one with radius 3438[1].

Figure 2 depicts a quadrant of the unit circle where $OX = OY = 1$. The arcs XA, XB and XC trace angles θ, $\theta + \phi$ and $\theta - \phi$ respectively. The half-chords AP, BQ and CR are the corresponding sine functions. We drop a perpendicular CS from the circumference to the half-chord BQ as shown. According to his commentator Nilakantha Somaiyaji [3], Aryabhata obtained the relationship between the difference in the trigonometric functions by demonstrating that $\triangle BSC$ and $\triangle OPA$ are similar and exploiting this property in an ingenious manner. We trace his line of reasoning in the Appendix where we derive

$$\sin(\theta + \phi) - \sin(\theta - \phi) = 2\sin(\phi)\cos\theta. \tag{1}$$

The difference in the sines is thus proportional to the cosine of the mean angle.

$$\cos(\theta + \phi) - \cos(\theta - \phi) = -2\sin(\phi)\sin\theta \tag{2}$$

The difference in the cosines is thus proportional to the (negative) of the sine of the mean angle.

1 A convention adopted by earlier workers. Note that one radian is 3438 minutes; 2π radians is 180°; and 1° is 60 minutes.

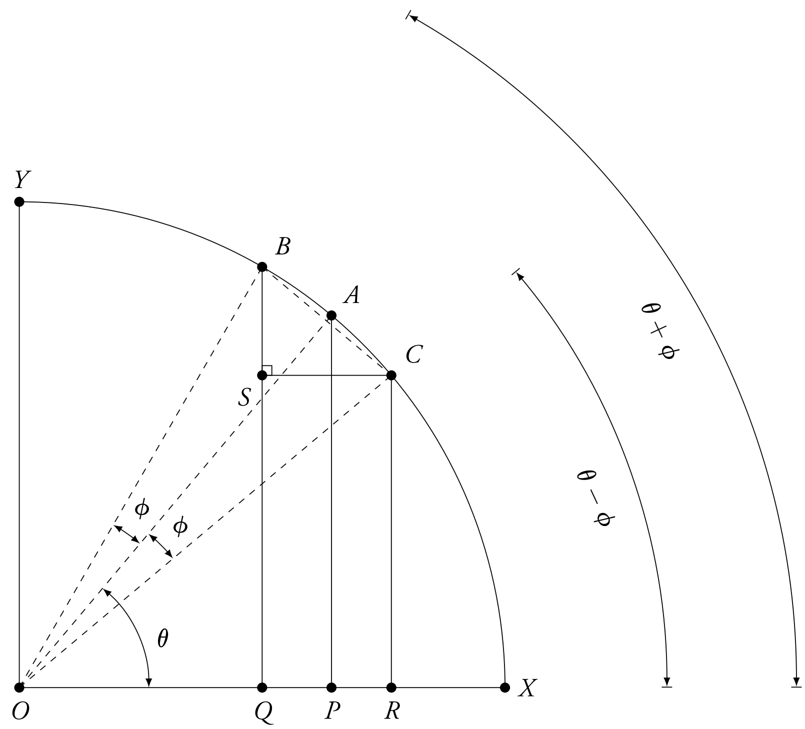

Figure 2. Derivation of the sine difference relation. The figure depicts the quadrant of a unit circle of radii $OX = OY = 1$. The half-chords AP, BQ and CR are $\sin\theta$, $\sin(\theta+\phi)$ and $\sin(\theta-\phi)$ respectively. It is worth noting that (later) we shall take ϕ to be a small angle.

Expressing the difference relations in another way, we get

$$\sin(x) - \sin(y) = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right), \text{ and} \tag{3}$$

$$\cos(x) - \cos(y) = -2\sin\left(\frac{x-y}{2}\right)\sin\left(\frac{x+y}{2}\right) \tag{4}$$

The Sine Table

Aryabhata begins with dividing the quadrant of the unit circle into 24 equal parts. He then obtains the values of the sines at fixed angles between 0 and $\pi/2$ thus generating the sine table for $\pi/48 = 3.75°$, $2\pi/48 = 7.5°$, $3\pi/48 = 11.25°$, …up to 90° (See Table 2). This table is given in verse 12 of the *Gitikapada*. It has been used by Indian astronomers (and astrologers!) in some form or another from 499 CE to the present. We shall see now how the table was generated.

As the quadrant is divided into 24 equal parts, let us take $\varepsilon = \frac{\pi}{48}$. Let us take $\phi = \varepsilon/2$ where ε is small. We now compute the differences of successive values of sine and cosine for the increment of ε. In other words, we compute the differences δs_n and δc_n defined below.

$$\delta s_n := \sin n\varepsilon - \sin(n-1)\varepsilon, \text{ and}$$
$$\delta c_n := \cos n\varepsilon - \cos(n-1)\varepsilon$$

where n varies from 0 to 24.

Using the equations (3) and (4), we get

$$\begin{aligned}\delta s_n &= \sin n\varepsilon - \sin(n-1)\varepsilon \\ &= 2\sin\left(\frac{n\varepsilon-(n-1)\varepsilon}{2}\right)\cos\left(\frac{n\varepsilon+(n-1)\varepsilon}{2}\right) \\ &= 2\sin\left(\frac{\varepsilon}{2}\right)\cos\left(\left(n-\frac{1}{2}\right)\varepsilon\right) \qquad (5)\end{aligned}$$

and

$$\begin{aligned}\delta c_n &= \cos n\varepsilon - \cos(n-1)\varepsilon \\ &= -2\sin\left(\frac{n\varepsilon-(n-1)\varepsilon}{2}\right)\sin\left(\frac{n\varepsilon+(n-1)\varepsilon}{2}\right) \\ &= -2\sin\left(\frac{\varepsilon}{2}\right)\sin\left(\left(n-\frac{1}{2}\right)\varepsilon\right) \qquad (6)\end{aligned}$$

We introduce the shorthand s_α to write $\sin(\alpha\varepsilon)$ and c_α to write $\cos(\alpha\varepsilon)$ and restate the equations (5) and (6) as

$$\delta s_n = s_n - s_{n-1} = 2s_{\frac{1}{2}}c_{(n-\frac{1}{2})} \qquad (7)$$

$$\delta c_n = c_n - c_{n-1} = -2s_{\frac{1}{2}}s_{(n-\frac{1}{2})} \qquad (8)$$

The above is a pair of coupled difference equations, and it was Aryabhata's insight to take the second difference, namely

$$\begin{aligned}\delta^2 s_n &= \delta s_n - \delta s_{n-1} = 2s_{\frac{1}{2}}\left(c_{(n-\frac{1}{2})} - c_{(n-\frac{3}{2})}\right) \\ &= -4s_{\frac{1}{2}}^2 s_{n-1} \qquad \text{from equation (8)} \qquad (9)\end{aligned}$$

Thus the second difference of the sines is proportional to the sine itself. The next step is to represent the RHS in terms of a recursion. We observe that s_n on the RHS of equation (9) may be written as

$$\begin{aligned}s_n &= s_n + (s_{n-1} - s_{n-1}) + (s_{n-2} - s_{n-2}) + \cdots + (s_1 - s_1) + (s_0 - s_0) \\ &= (s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) + \cdots + (s_1 - s_0) + s_0 \\ &= \delta s_n + \delta s_{n-1} + \cdots + \delta s_1 + 0 \\ &= \sum_{m=1}^{n} \delta s_m\end{aligned}$$

Thus

$$\delta^2 s_n = \delta s_n - \delta s_{n-1} = -4s_{1/2}^2 \sum_{m=1}^{n-1} \delta s_m \qquad (10)$$

Thus we get a recursion relation where the second difference of the sines is expressed in terms of all previously obtained first and second sine differences. To initiate the recursion, we need δs_1 which is $s_1 - s_0 = \sin\varepsilon - \sin 0 \approx \varepsilon$, since for small angles the half-chord and the corresponding arc are equal, as stated in the previous section.

Using the recursion relation we can generate Aryabhata's celebrated sine table, taking π = 3.1416 and $\sin\varepsilon = \varepsilon = 0.0654\ (= 225')$.

Table 2 depicts some typical values of the sine function as well as the value of the sine multiplied by 3438 (the so called '*Rsine*' of Aryabhata). We see that this matches Aryabhata's sine table to ± 1 minute. For example, $\theta = \pi/6$ gives 1719 minutes. For comparison, we also give the current accepted value of $\sin\theta$ to four decimal points. Note that Aryabhata takes angles only till $\pi/2$; he seems aware of the fact that going further is unnecessary given the periodic nature of the sine function.

θ	$\sin\theta$ (Aryabhata)	$\sin\theta$ (minutes)	$\sin\theta$ (modern)
$\pi/48$	0.0654	225	0.0654
$2\pi/48$	0.1305	449	0.1305
$3\pi/48$	0.1951	671	0.1951
$4\pi/48$	0.2588	890	0.2588
$5\pi/48$	0.3214	1105	0.3214
$6\pi/48$	0.3827	1315	0.3827
$7\pi/48$	0.4423	1520	0.4423
$8\pi/48$	0.5000	1719	0.5000
$9\pi/48$	0.5556	1910	0.5556
$10\pi/48$	0.6088	2093	0.6088
$11\pi/48$	0.6594	2267	0.6593
$12\pi/48$	0.7072	2431	0.7071
$13\pi/48$	0.7519	2585	0.7518
$14\pi/48$	0.7935	2728	0.7934
$15\pi/48$	0.8316	2859	0.8315
$16\pi/48$	0.8662	2978	0.8660
$17\pi/48$	0.8971	3084	0.8969
$18\pi/48$	0.9241	3177	0.9239
$19\pi/48$	0.9472	3256	0.9469
$20\pi/48$	0.9662	3322	0.9659
$21\pi/48$	0.9812	3373	0.9808
$22\pi/48$	0.9919	3410	0.9914
$23\pi/48$	0.9983	3432	0.9979
$24\pi/48$	1.0005	3439	1.0000

Table 2. Table of sine values using Aryabhata's method, $\varepsilon = \pi/48 = 3.75° = 225'$ and $\pi = 3.1416$ and comparison with modern day values. In column 3 we quote values in minutes as done in Verse 12 of the *Gitika* chapter of *Aryabhatiya* [1, 2].

Finite Difference Calculus

Of greater relevance is the fact that the sine (or cosine) difference formulae foreshadow finite difference calculus, a popular numerical technique in this age of computation. Rewriting equations (1) and (2) with $\phi = \varepsilon$,

$$\frac{\sin(\theta+\varepsilon)-\sin(\theta-\varepsilon)}{2\sin(\varepsilon)} = \cos\theta \tag{11}$$

$$\frac{\cos(\theta+\varepsilon)-\cos(\theta-\varepsilon)}{2\sin(\varepsilon)} = -\sin\theta \tag{12}$$

Aryabhata took ε to be $\pi/48$. But he also stated that its value is *yateshtani* or 'as per our wish' (Verse 11, *Ganitapada*). Some took it to be $\pi/96$; others (like Brahmagupta) took it as $\pi/12$ or 15°. If we take ε to be sufficiently small, we have our classic formula for finite difference calculus. Noting that $2\sin\frac{\varepsilon}{2} \approx \varepsilon$ we have the finite difference version of the derivative of sine,

$$\frac{\sin\theta}{d\theta} = \cos\theta,$$

and similarly for the cosine,

$$\frac{\cos\theta}{d\theta} = -\sin\theta.$$

Let us illustrate this with an example. We know that $\sin 37^\circ \approx 0.6$, and $\sin 30^\circ = 0.5$. The difference in angle is 7° which in radians is 0.122. Thus the derivative of sine of the mean angle 33.5° from equation (11) is

$$\delta\sin\theta/\delta\theta = (0.6-0.5)/0.122 = .82.$$

Looking up the sine table or the calculator yields $\cos 33.5^\circ = 0.83$. Similarly equations (9) yields the second derivatives namely

$$\delta^2\sin\theta/\delta^2\theta \approx -\sin\theta$$
$$\delta^2\cos\theta/\delta^2\theta \approx -\cos\theta$$

The above are now called central difference approximations to the first derivative and the second derivative. Naturally, Aryabhata does not use the term 'finite difference calculus' (or 'calculus'). But similar methods are now used to numerically solve differential equations. The student will recognize the above as a standard solution of the classical simple harmonic oscillator. We note in passing that Newton's II Law and the famous Schrödinger equation of quantum mechanics are both second-order differential equations.

Discussion

One can discern a continuity in Indian mathematics, however tenuous, from pre-Vedic times (prior to 1000 BCE) till the 1800s. A striking example is the influence of *Aryabhatiya* on major Indian mathematicians who followed him including Madhava (1350 CE) who founded Calculus [4]; as also the influence on Aryabhata of the mathematics which preceded him [6, 7].

To reiterate, Aryabhata seems aware that (i) $\sin 0^\circ = 0$; (ii) the sine of a small angle is itself, as the small arc is 'almost equal' to the half-chord; (iii) $\sin 30^\circ = 1/2$ (*Ganitapada* verse 9); (iv) $\sin 90^\circ = 1$; (v) the sine function is periodic, so he prudently does not extend the computation to angles greater than 90°. Then, in a remarkably insightful way, he lays down the recursion relation for sine differences which enables one to generate the sine table. It is this work, more than his solution to the linear Diophantine equation (verses 31 and 32 of *Ganitapada*) which establishes him as a genius and one of the brightest stars in the firmament of world mathematics.

The sine table can also be generated using the half-angle formula. This was demonstrated in the *Panchasiddhantika*, a text written barely 50 years after the appearance of *Aryabhatiya* [8]. As pointed out, a feature of the Aryabhata's difference relation is how modern it is. It can readily be seen to be essentially the same as finite difference calculus. It led to the development of the calculus of trigonometric functions by Madhava (1350 CE) and his disciples along the banks of the Nila river in Kerala. This school is variously called the Nila [4] and the Aryabhata school [7]. Another aspect to note is that Bhaskara II (1100 CE) used the division of the great circle into parts of magnitude $2\pi/96$ to carry out discrete integration and obtain the (correct) expressions for the surface area and the volume of a sphere. Jyesthdeva of the Nila (or Aryabhata) school in his work *Yuktibhasa* derived the same results using calculus (circa 1500 CE). Aryabhata can thus legitimately be called the founder of trigonometry.

To sum up, the *Aryabhatiya* exercised a tremendous influence over Indian mathematicians for over a thousand years. For a book with just over one hundred pithy verses, its legacy remains unparalleled in the scientific world. We hope that our article will give our young audience an introduction to his work and will serve as an inspiration.

Acknowledgement. One of the authors (VAS) would like to place on record the many useful discussions he had with Prof. P. P. Divakaran.

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Appendix: The Derivation of the sine and cosine difference formula

We add Figure 2 once again here (labelled as Figure 3) for the reader's quick reference. As stated in the text the figure depicts a quadrant of the unit circle where $OX = OY = 1$. The arcs XA, XB and XC trace angles θ, $\theta + \phi$ and $\theta - \phi$ respectively. The half-chords AP, BQ and CR are the corresponding sine functions. We drop a perpendicular CS from the circumference onto the half-chord BQ as shown.

We show that $\triangle BSC$ and $\triangle OPA$ are similar. By construction $\measuredangle BSC$ and $\measuredangle OPA$ are each $90°$. Note $OB = OC = 1$ (unit radius) and hence $\triangle OBC$ is isosceles. This implies that

$$\measuredangle OBC = \measuredangle OCB = \frac{180° - 2\phi}{2} = 90° - \phi.$$

Also in $\triangle OBQ$,

$$\measuredangle OBQ = 180° - \measuredangle OQB - \measuredangle BOQ = 180° - 90° - (\phi + \theta) = 90° - \phi - \theta.$$

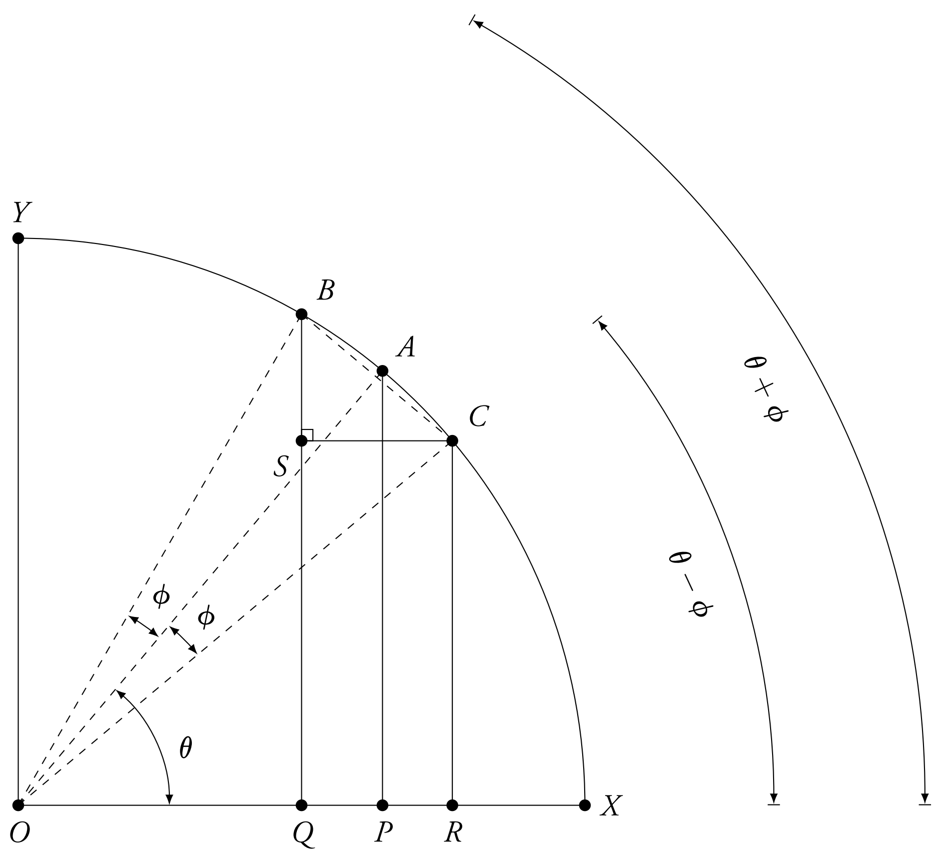

Figure 3. Derivation of the sine difference relation.

Hence $\measuredangle SBC = \measuredangle OBC - \measuredangle OBQ = \theta$. Therefore, $\measuredangle SBC = \measuredangle POA = \theta$. This establishes the similarity of the two triangles by the angle-angle test.

$$\frac{BS}{OP} = \frac{BC}{OA}$$

Equivalently,

$$BS = \frac{BC}{OA} OP \tag{13}$$

On LHS of (13), we have

$$BS = BQ - CR = \sin(\theta + \phi) - \sin(\theta - \phi).$$

On the other hand, on RHS of (13), we have $OA = 1$ (unit radius) and $OP = \cos\theta$. Since $BC \perp OA$, we also have $BC = 2\sin(\phi)$. This yields the sine difference formula (equation (1))

$$\sin(\theta + \phi) - \sin(\theta - \phi) = 2\sin(\phi)\cos\theta.$$

Thus the difference in the sines is proportional to the cosine of the mean angle. We can also obtain the cosine difference formula (equation (1) by noting that

$$\frac{CS}{AP} = \frac{BC}{OA}.$$

Note $AP = \sin\theta$ and $CS = OR - OQ = \cos(\theta - \phi) - \cos(\theta + \phi)$. Hence

$$\cos(\theta + \phi) - \cos(\theta - \phi) = -2\sin(\phi)\sin\theta.$$

The difference in the cosines is proportional to the (negative) of the sine of the mean angle. We pause to note that *prima facie* the two triangles we considered appear unrelated. A hallmark of Indian mathematics is strong geometric intuition and this dates back to the *Sulbasutra circa* 800 BCE. Another is the reliance on the 'rule of three' (*trirasikam*). Here we employ a simple version of it namely, if $a/b = c$ then $a = b \times c$.

Exercises

(1) We can generate the sine table as per Aryabhata's suggestion but not using his value for ε. We choose $\varepsilon = \pi/80 \approx 0.0393$ which is the same as 2.25°. We take $\sin(\varepsilon) \approx \varepsilon$. If you have a calculator, generate all values of sine from 2.25° to 18° in equal steps using equation (10).
Alternatively, if you have a programmable calculator or a computer, generate all values of sine from 2.25° to 90°. Compare with the results your calculator yields.

(2) In the last section, reference is made of the text *Panchasiddhantika* wherein the half angle formula is mentioned:

$$\cos(2\theta) = 1 - 2\sin^2\theta.$$

How would you (i) derive this by a geometrical construction; (ii) employ this to generate the sine table?

Problem Corner

Detecting poisoned samples: A computational thinking activity using binary arithmetic

KUMAR GANDHARV MISHRA

Introduction

Picture yourself as a lab technician during a pandemic, faced with a critical challenge. Resources are scarce, and you need to identify which one of the 15 samples is contaminated with a virus. The catch? You have only 4 test tubes at your disposal, and you want to filter out negative samples using only these 4 test tubes in the most efficient manner. Can you detect the 'poisoned' sample?

This article is based on a hypothetical simulation activity to detect poisoned samples from a set of given bottles. What makes the situation interesting is that the number of samples is greater than the number of test tubes available for detecting poisoned samples. The simulation can be done either using spreadsheets or pen-and-paper, and provides an opportunity to learn binary arithmetic and explore different situations related to detecting samples.

Computational Thinking in the Classroom

The National Education Policy 2020 and the National Curriculum Framework for School Education 2023 emphasise computational thinking in the classroom. It has been described as one of the key 21st century skills in daily life. In simple words (see [1]), computational thinking involves describing a problem; identifying the important details needed to solve this problem; breaking down the problem into small and logical steps; using

Keywords: Binary arithmetic, Computational thinking, Reasoning, Sample sorting

these steps to create a process that solves the problem; and evaluating this process. It uses algorithmic thinking, abstraction, decomposition, pattern recognition and generalisation, evaluation and logic to solve the problem at hand.

Computational thinking is also about thinking like a computer scientist, and when computers come in, how can binary numbers be left behind? Computers store information in the form of binary digits (bits), and understanding binary representation can help us understand the workings of computers because, at a fundamental level, computers are just machines for flipping binary digits on and off. Teaching binary numbers as an introduction to computational thinking introduces students to algorithms and decomposition, and during this process, they learn to break down the problems of calculating binary numbers and converting between binary and decimal numbers, into step-by-step processes. It also introduces abstraction, as students learn that two different things (0 and 1) can be used to represent any and all information (See [2]).

Keeping these points in mind, this article outlines an activity that involves distinguishing between two types of samples—positive and negative—within a larger set. During this activity, learners engage in computational thinking by using binary numbers to represent different samples and breaking down the problem into small steps. They apply reasoning and logic to determine the status of each sample based on the colour patterns displayed in test tubes. It also involves playing with computers using spreadsheets on Excel.

About the activity

For simplicity, let's assume we have a chemical indicator that reveals viral contamination by changing the colour of the sample. If the sample turns red, it indicates that the sample is poisoned, and we describe this result as 'positive.' Conversely, if the sample turns green, it shows that the sample is not poisoned, which we refer to as 'negative.' (Note that in this context, 'negative' means the sample is safe and unpoisoned).

Given that the number of test tubes is less than the number of samples, our goal is to efficiently filter out the negative samples, thereby reducing the testing burden significantly in the first step. Since four test tubes are insufficient to simultaneously test each of the 15 samples individually, the samples must be mixed within these four tubes. We capitalise on the fact that we have fifteen samples and four test tubes, By employing binary arithmetic, we can code the samples in such a way that allows us to identify those likely poisoned. Table 1 below shows the binary equivalents of decimal numbers up to 15. Using the binary coding system the samples can be represented as:

Sample No	Binary representation	Sample No	Binary representation
$1 = 2^0$	0001	$9 = 2^3 + 2^0$	1001
$2= 2^1$	0010	$10 = 2^3 + 2^1$	1010
$3 = 2^1 + 2^0$	0011	$11 = 2^3+ 2^1 + 2^0$	1011
$4 = 2^2$	0100	$12 = 2^3 + 2^2$	1100
$5 = 2^2 + 2^0$	0101	$13 = 2^3 + 2^2 + 2^0$	1101
$6 = 2^2 + 2^0$	0110	$14 = 2^3 + 2^2 + 2^1$	1110
$7 = 2^2 + 2^1 + 2^0$	0111	$15 = 2^3 + 2^2 + 2^1 + 2^0$	1111
$8 = 2^3$	1000		

Table 1: Reading from right to left, a 0 is put where there is no power of 2 and a 1 where there is.

Each binary code **also** represents a unique way of putting a sample in these four test tubes according to its 4-digits. Let '1' denote putting a sample into the corresponding test tube and a '0' denote skipping the test tube. Let us denote the test tubes as T1, T2, T3 and T4. Now, for example, sample number 5, whose binary representation is 0101, can be put into the test tubes T2 and T4 (1 is at the 2^{nd} and 4^{th} position in binary code of 5), while sample number 6 (0110) can be put into T2 and T3 and so on. Each sample can be put in at least one test tube and at most 4 test tubes. For instance, sample number 1 will be put in only one test tube, and sample number 15 will be put in all 4 test tubes. If a test tube turns green, it means all samples put in this tube are negative. On the other hand, if a test tube turns out red then it doesn't mean that **all** samples related to it are positive, that test tube can turn red even due to **one** positive sample put in it. So, in a way, the green signal helps us filter all those samples which were put in that test tube.

An Illustration

Let's take a case where two of the test tubes indicate at least one positive sample in them.

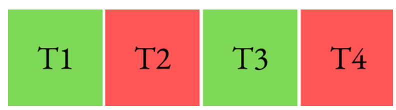

Figure 1: Results obtained in 4 test tubes.

Here test tubes T1 and T3 turn green, while T2 and T4 turn red. This result conclusively identifies all samples placed in either T1 or T3, or both, as negative, given that the binary '1's for these samples correspond to the green test tubes. Referring to Table 1, we can conclude that :

- T1 is green: All samples placed in T1 are negative, which corresponds to samples that have a binary '1' in the first position—specifically, samples 8 to 15.
- T3 is green: Similarly, all samples placed in T3 are negative, corresponding to samples with a binary '1' in the third position—samples 2, 3, 6, 7, 10, 11, 14 and 15.

On the other hand, the samples placed in T2 and T4 are suspected of being positive. However, it doesn't mean that each sample put into these test tubes is positive. Each sample put in T2 (4, 5, 6, 7, 12, 13, 14 & 15) can be positive or at least one is positive for sure, the same goes for T4 (1, 3, 5, 7, 9, 11, 13, 15). Comparing with the conclusions from the green test tubes, we filter out the negative samples (3, 6, 7, 9, 11, 12, 13 14, 15) and reduced our task to identify the provisionally (those samples which are considered to be positive unless and until proven negative) positive samples (1, 4, 5), which we can re-test.

Some noteworthy points: The test tube colour combination shouldn't be related to similar appearing binary code and the respective decimal number. For example, the pattern of the test tube for positive result in Figure 1 appears as 0101 which matches with the code of sample number 5, but sample number 5 may be negative. T2 and T4 can show positive results not only because of sample 5, but also because of other samples which have been put in these test tubes. For example, one positive sample in T2 and one positive sample in T4 can make the test tube combination positive, while sample number 5 could be negative.

Think logically and reason out!

Suppose according to the test results T4 is green and T1, T2 and T3 are red.

What would be your conclusion?

Activity in the classroom

The teacher can convert the above binary-based reasoning into an in-class activity involving a series of tasks.

1. The teacher introduces the problem at hand and provides an illustration.
2. The students are asked to fill the following table depicting the binary representations of sample numbers and which of the 15 samples are put into each of the 4 test tubes. (We illustrate for sample number 6 here)

Sample No	Binary representation	Test tubes			
		T1	T2	T3	T4
1	0001				
2	0010				
3	0011				
4	0100				
5	0101				
6	0110	✕	✓	✓	✕
7	0111				
8	1000				
9	1001				
10	1010				
11	1011				
12	1100				
13	1101				
14	1110				
15	1111				

Table 2

3. The students are asked to count how many combinations of test results are possible.
4. The students are divided into pairs and assigned a few particular test results (by lottery) given as coloured strips (two distinguishable colours – say, red and green). They are instructed to try to filter out the negative samples based on the test tube colour combinations they got.
5. The teacher demonstrates for a sample. If the given colour strip has T1 and T3 in red colour, and T2 and T4 in green colour as in Figure 3, the teacher explains how to filter out the negative samples and record them in the table generated by the students (Figure 2).

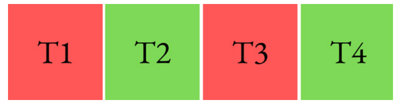

Figure 2

Sample No	Binary representation	Test tubes			
		T1	T2	T3	T4
1	0001	✗	✗	✗	✓
2	0010	✗	✗	✓	✗
3	0011	✗	✗	✓	✓
4	0100	✗	✓	✗	✗
5	0101	✗	✓	✗	✓
6	0110	✗	✓	✓	✗
7	0111	✗	✓	✓	✓
8	1000	✓	✗	✗	✗
9	1001	✓	✗	✗	✓
10	1010	✓	✗	✓	✗
11	1011	✓	✗	✓	✓
12	1100	✓	✓	✗	✗
13	1101	✓	✓	✗	✓
14	1110	✓	✓	✓	✗
15	1111	✓	✓	✓	✓

Table 3

In this case, the negative samples are 1, 3, 4, 5, 6, 7, 9, 11, 12, 13, 14 and 15. The teacher then proceeds to explain how to find the positive samples. In this case, the only remaining samples are 2, 8 and 10. These samples can be renumbered as I, II, and III respectively and tested again using test tubes T1, T2 and T3 alone. The students are asked to draw possible outcomes if only one of these samples is poisoned.

6. The students work in pairs and record their results. In the middle of the activity, any two groups which got the same test result could share their conclusions. At the end of the activity, the teacher randomly picks a few pairs and asks them to share their findings with the class.

7. The students are provided with a worksheet to work on further.

Suggestions:

To make the activity tactile, the students can divide themselves into pairs and generate all 16 possible test results as colour patterns (as in Figure 4) and put them in a bowl. Each pair can draw one cutout from the bowl and do the prescribed activity. This activity reinforces the computational thinking aspects by decomposing the problem into small steps and using logic detecting the negative samples and probable positive samples.

Try yourself:

Suppose T1, T2, T3 turn green, and T4 turns red (Figure 3).

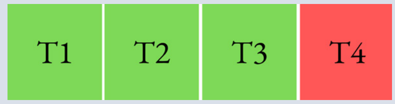

Figure 3

How would you proceed?

Let's brainstorm!

1. Suppose that sample number 15 is poisoned. What would be the result you should expect? Indicate the colours of the test tubes.
2. If sample number 15 is poisoned, how will you identify it?
3. If you have five test tubes, up to how many samples can you test? What about six test tubes?
4. How many test tubes are required to test 1000 samples? Why?
5. In the following situations, calculate the number of provisionally positive (Those samples which are probable of being positive i.e. positive unless and until proved to be negative) samples.

Number of green-coloured test tubes	Number of red-coloured test tubes	Provisionally positive samples
0	4	?
1	3	?
2	2	?
3	1	?
4	0	?

Table 4

Converting the reasoning task into an activity using a simulation

This activity can be transformed into a simulation activity for students using Excel spreadsheets.

Let's explore the different colour permutations of test tubes through Excel and the random function generator:

1. Open an Excel file spreadsheet and select any cell.

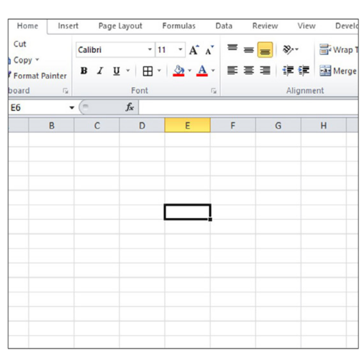

Figure 4

2. Use '=CHOOSE (RANDBETWEEN (1,2), "positive", "negative")' function at the cell. With the selection key on the cell, expand the selection up to four cells horizontally. The terms 'positive' and 'negative' will appear in these four cells. Consider these 4 cells as 4 test tubes.

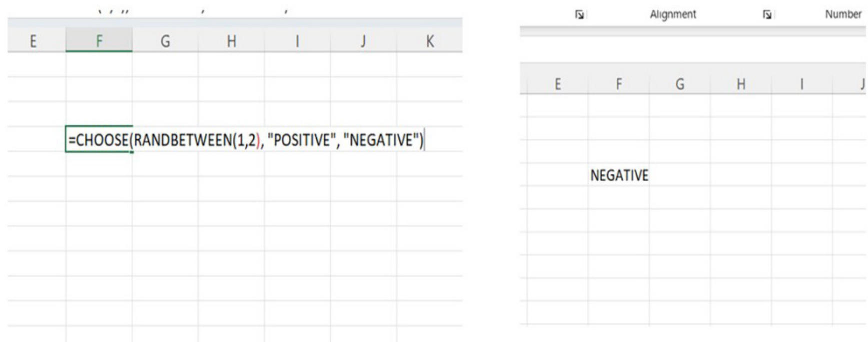

Figure 5

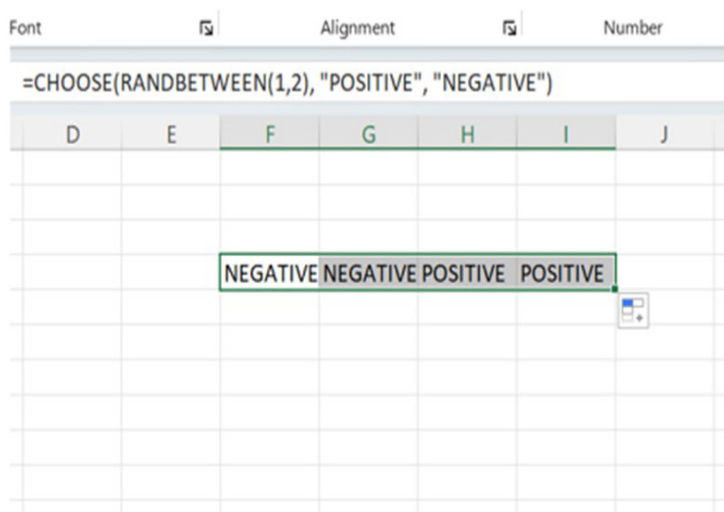

Figure 6

3. Select a cell other than these cells. Write any letter or number in this cell and press enter. You will observe that the words in the four cells change. Repeat this process by entering any number in any other cell; again, you will observe that the terms positive and negative vary in these 4 cells.

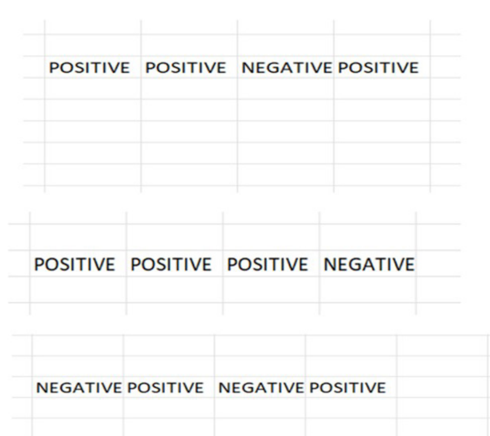

Figure 7

4. Now, select the 4 cells (test tubes) and using conditional formatting (highlight the cell that contains a specific text) label the cells with red colour and green colour (red for positive and green for negative).

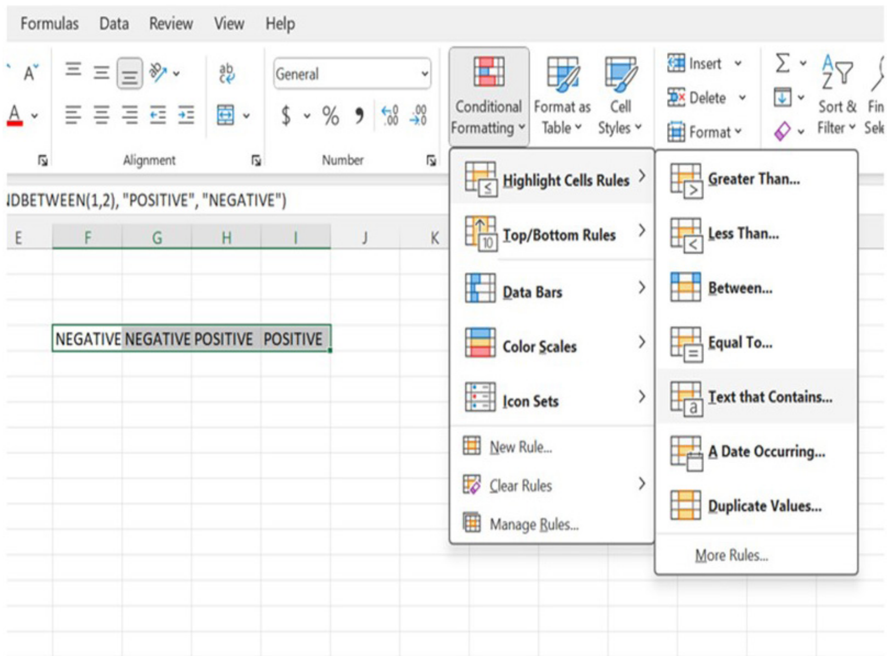

Figure 8

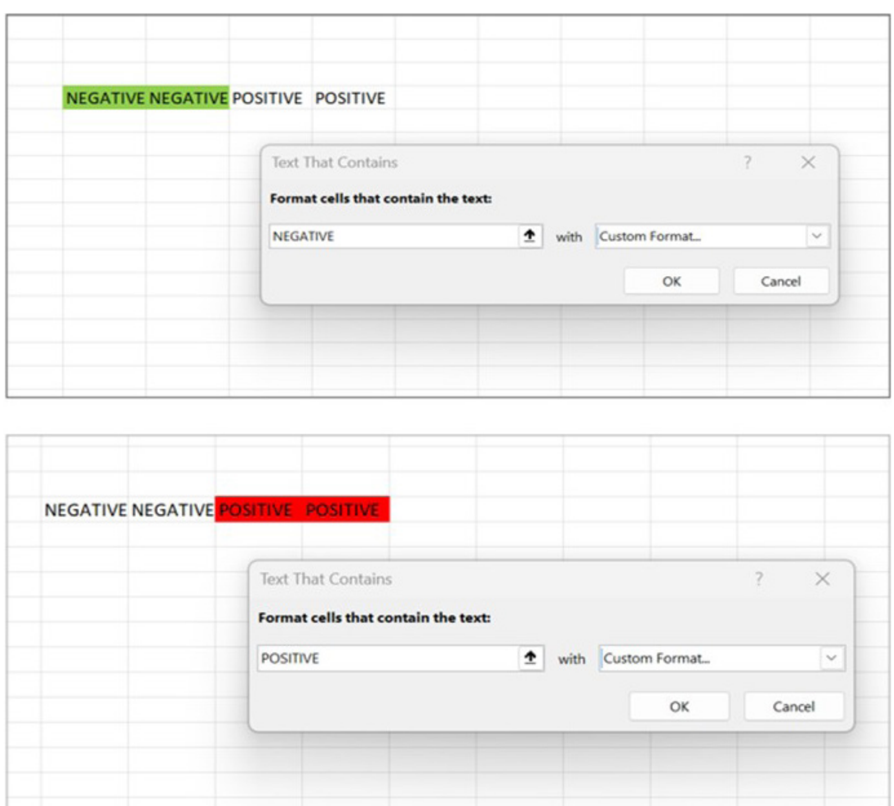

Figure 9

5. Reduce the font size of terms ‘positive’ and ‘negative’ as small as possible; say up to size 1 so that only colours appear on the test tube cells and the numbers get diminished.

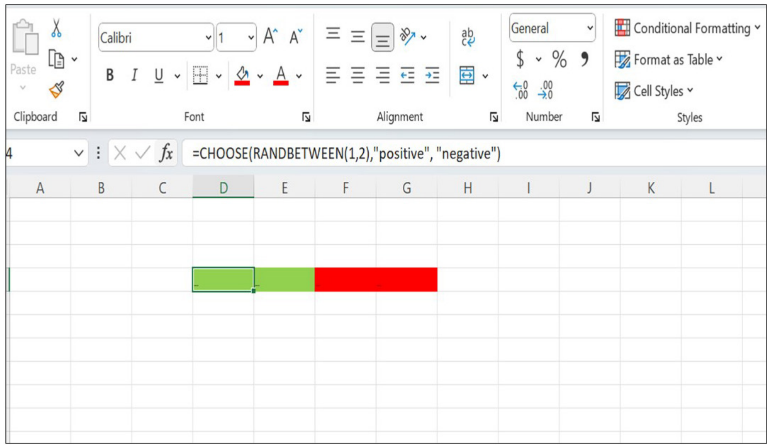

Figure 10

6. Select any other cell and write any number and press enter, the colour of the test tube cell changes. By doing this you can get different permutations of colour in the test tube cells. Some examples of possible permutations:

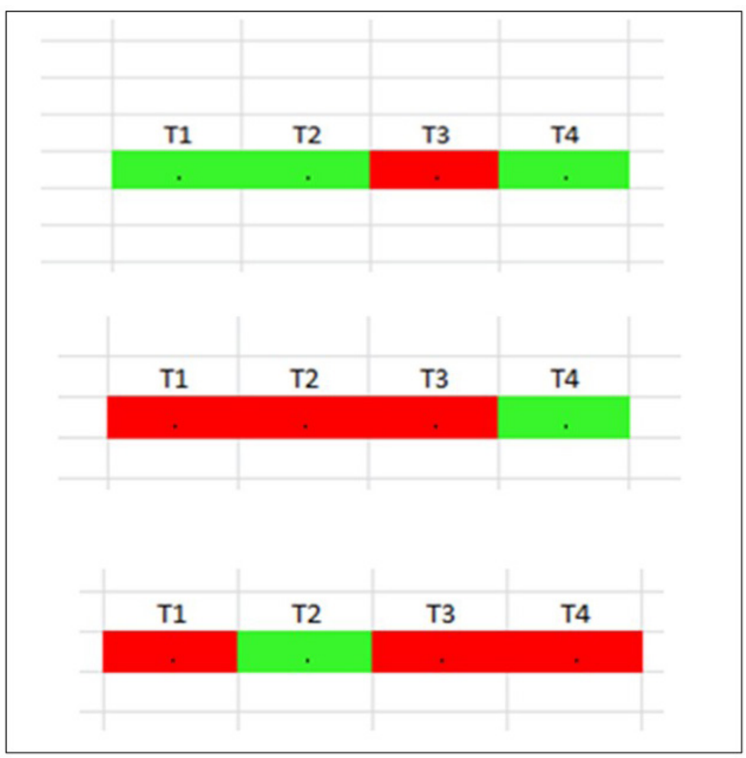

Figure 11

7. Consider the red cells as an indicator of positive/poisoned (provisional) samples put in them and green cells as an indicator of negative samples put in them. So, a total of 2^4 i.e., 16 colourful permutations can be generated. Among these 16 permutations, one of the permutations can refer to all 4 test tubes with zero i.e., skipping all test tubes. So, we don’t need to focus on this situation, but on the rest of the 15 situations.

8. Create the table using spreadsheets, depicting the relationship between 15 samples and 4 test tubes.

		Test Tubes			
Sample Number	Binary Identity	T1	T2	T3	T4
1	0001	0	0	0	1
2	0010	0	0	1	0
3	0011	0	0	1	1
4	0100	0	1	0	0
5	0101	0	1	0	1
6	0110	0	1	1	0
7	0111	0	1	1	1
8	1000	1	0	0	0
9	1001	1	0	0	1
10	1010	1	0	1	0
11	1011	1	0	1	1
12	1100	1	1	0	0
13	1101	1	1	0	1
14	1110	1	1	1	0
15	1111	1	1	1	1

Figure 12

Now, we will use the same reasoning as shown in Figure 12.

Conclusion

During the whole process, students use reasoning and logic to decompose the task of distinguishing the samples (positive or negative as red or green) by filtering out negative samples and again working with the smaller lot. The student engages with binary arithmetic when he/she gives identity to each sample, looks for patterns among the test tubes and evaluates his/her decision of labelling a sample as negative or provisionally positive. This is what computational thinking is all about in this activity, where the teacher provides an opportunity for the student to decompose the problem into small steps with logic and devise his/her algorithm to solve while playing with binary numbers.

Author's note: A similar approach to test Covid-positive samples through binary arithmetic also appeared as a suggestion by Kadri (2020), but it used the concept of ratio of mixing of samples.

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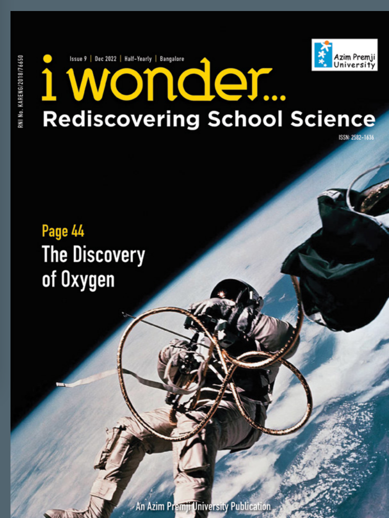

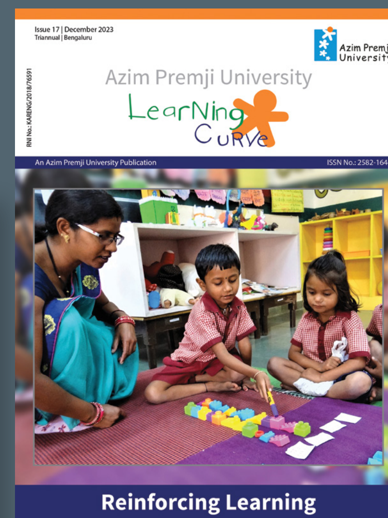

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